On the Converse to a Theorem of Atiyah and Bott

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Introduction

Throughout this paper, C denotes a smooth projective curve of genus at least one, G denotes a reductive linear algebraic group over $\mathbb{C}$, and ξ_0 is a C^∞ principal G-bundle over C. The space of all $(0,1)$-connections on ξ_0 is an affine space $\mathcal{A} = \mathcal{A}(\xi_0)$ associated to the infinite dimensional complex vector space $H^{0,1}(C; \operatorname{ad} \xi_0)$. Following Shatz [6] for the case $G = GL(n)$, Atiyah and Bott [1] defined a natural stratification of this space. If $\mathfrak{h}$ is a Cartan subalgebra for G, then the strata are indexed by the orbits under the Weyl group of a certain discrete set of points in $\mathfrak{h}_{\mathbb{R}}$, the split real form of $\mathfrak{h}$, which we call points of *Atiyah-Bott type* for ξ_0. Fix a set Δ of simple roots for G with respect to $\mathfrak{h}$. Since every Weyl orbit in $\mathfrak{h}_{\mathbb{R}}$ has a unique representative in the positive Weyl chamber $\overline{C}_0$ associated to Δ, it is natural to index the strata by points μ of Atiyah-Bott type for ξ_0 which lie in $\overline{C}_0$. We denote by $\mathcal{C}_\mu$ the stratum corresponding to μ. A point μ of Atiyah-Bott type for ξ_0 determines a parabolic subgroup $P(\mu)$ of G, together with a C^∞ $P(\mu)$-bundle $\eta_0(\mu)$ over $P(\mu)$ such that $\eta_0(\mu) \times_{P(\mu)} G$ is C^∞ isomorphic to ξ_0. The condition that $\mu \in \overline{C}_0$ is just the condition that $P(\mu)$ is a standard parabolic subgroup. Recall that an unstable holomorphic bundle ξ whose underlying C^∞-bundle is C^∞ isomorphic to ξ_0 has a canonical reduction to a parabolic subgroup, called the *Harder-Narasimhan* reduction. A $(0,1)$-connection lies in $\mathcal{C}_\mu$ if and only if the parabolic subgroup of its Harder-Narasimhan reduction is conjugate to $P(\mu)$ in such a way that the corresponding holomorphic $P(\mu)$-bundle is C^∞ isomorphic to $\eta_0(\mu)$. The strata $\mathcal{C}_\mu$ are of finite codimension in $\mathcal{A}$ and are invariant under the action of the group of C^∞-changes of gauge. We define $\mathcal{C}_{\mu'} \preceq_r \mathcal{C}_\mu$ if the closure of $\mathcal{C}_{\mu'}$ meets $\mathcal{C}_\mu$, and define the relation $\preceq$ on the strata $\mathcal{C}_\mu$ by taking the unique extension of $\preceq_r$ to a transitive relation. More concretely, $\mathcal{C}_{\mu'} \preceq \mathcal{C}_\mu$ if and only if there is a sequence $\mathcal{C}_{\mu'} = \mathcal{C}_{\mu_0}, \mathcal{C}_{\mu_1}, \dots, \mathcal{C}_{\mu_n} = \mathcal{C}_\mu$ such that for each i, $0 \le i \le n-1$ the closure of $\mathcal{C}_{\mu_i}$ meets $\mathcal{C}_{\mu_{i+1}}$.

For Atiyah and Bott, the stratification arises as follows. Let K be a compact Lie group whose complexification is G, and let ξ_K be a C^∞ principal K-bundle such that $\xi_K \times_K G$ is C^∞ isomorphic to ξ_0. One can then identify K-connections on ξ_K with $(0,1)$-connections

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on ξ_0. Under this identification, the strata $\mathcal{C}_\mu$ are the stable sets under the negative gradient flow for the Yang-Mills functional $\|F_A\|^2$ on the space of K-connections on ξ_K for the various connected components of the critical set of the Yang-Mills functional. From this point of view, $\mathcal{C}_{\mu'} \preceq_r \mathcal{C}_\mu$ if there is a downward gradient flow line in the stratum $\mathcal{C}_{\mu'}$ converging as $t \to -\infty$ to a point in $\mathcal{C}_\mu$.

Atiyah and Bott introduce a partial ordering, the *Atiyah-Bott ordering*, on points $\mu \in \mathfrak{h}$, as follows: $\mu' \leq \mu$ if and only if the convex hull of the Weyl orbit of μ contains μ'. They then prove:

Theorem. *Let μ, μ' be points of Atiyah-Bott type for ξ_0. If the closure of a stratum $\mathcal{C}_{\mu'}$ meets another stratum $\mathcal{C}_\mu$ then $\mu' \leq \mu$ in the Atiyah-Bott ordering.*

In particular, it follows that the relation $\preceq$ is a partial ordering on the set of strata.

The purpose of this paper is to prove a converse to this theorem, by showing:

Theorem. *Let ξ_0 be a C^∞-principal G-bundle over C. Suppose that μ, μ' are points of Atiyah-Bott type for ξ_0. If $\mu' \leq \mu$, then $\mathcal{C}_{\mu'} \preceq \mathcal{C}_\mu$. Thus, $\mathcal{C}_{\mu'} \preceq \mathcal{C}_\mu$ if and only if $\mu' \leq \mu$ in the Atiyah-Bott partial ordering.*

In general, the decomposition $\{\mathcal{C}_\mu\}_\mu$ of the space of $(0,1)$-connections is not a stratification in the sense that the closure of a stratum is not a union of the higher strata with respect to the Atiyah-Bott ordering. There is only an inclusion. In Section 4.3 we give an example for the group $SL(3)$ and any curve of genus greater than one to show that this inclusion is not in general an equality. One could ask if a somewhat stronger statement than the above theorem holds: given $\mu' \leq \mu$, does there exist a point of $\mathcal{C}_\mu$ which is in the closure of $\mathcal{C}_{\mu'}$? It is possible that the techniques of this paper can be extended to prove this somewhat stronger statement.

For curves of genus one the situation is much better: in this case we have a stratification in the strong sense.

Theorem. *Suppose that $g(C) = 1$, and fix a C^∞ principal G-bundle ξ_0 over C. Let μ be a point of Atiyah-Bott type for ξ_0. Then the closure of the stratum $\mathcal{C}_\mu$ in the space of $(0,1)$-connections on ξ_0 is the union $\bigcup_{\mu' \geq \mu} \mathcal{C}_{\mu'}$ where μ' ranges over points of Atiyah-Bott type for ξ_0.*

Our motivation for this work came out of the study of the moduli space of semistable G-bundles over an elliptic curve. One can describe this moduli space as a space of all nontrivial deformations of a "minimally unstable" G-bundle, which makes clear its structure as a weighted projective space. For many cases (but not for $SL(n)$), it turns out that there is in fact essentially a unique such minimally unstable bundle. This paper is our attempt to understand why this should be so.

The contents of this paper are as follows. In Section 1, we collect some preliminaries on root systems and parabolic subgroups. In Section 2, we define points of Atiyah-Bott type and discuss the Harder-Narasimhan reduction, the strata, and the Atiyah-Bott ordering. Section 3 deals with what we call harmonic and superharmonic functions on a Dynkin

diagram, which are convenient ways to record some of the positivity properties of the Cartan matrix. In Section 4, we formulate the main theorems as a combinatorial problem and show how this problem can be translated into a set of results about bundles. Most of these results concerning bundles are also established there. However, one such result requires the notion of an elementary tranformation of G-bundles. This is defined in Section 5, and we then prove the relevant bundle result. Finally, in Section 6 we study the problem of finding minimal elements in the Atiyah-Bott ordering such that the strata correspond to unstable bundles.

1 Preliminaries

1.1 Basic notation

We denote by $\mathfrak{g}$ the Lie algebra of G, by $Z(G)$ the center of G and by $\mathfrak{z}_G$ the center of $\mathfrak{g}$. Fix a maximal torus H for G with associated Cartan subalgebra $\mathfrak{h}$, so that $\mathfrak{z}_G \subseteq \mathfrak{h}$. There is a direct sum decomposition $\mathfrak{h} = (V^* \otimes \mathbb{C}) \oplus \mathfrak{z}_G$. If K is a maximal compact subgroup of G, we let $\mathfrak{g}_{\mathbb{R}} = \sqrt{-1}\,\mathrm{Lie}\,K$, and define $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h} \cap \mathfrak{g}_{\mathbb{R}}$, $(\mathfrak{z}_G)_{\mathbb{R}} = (\mathfrak{z}_G) \cap \mathfrak{g}_{\mathbb{R}}$. Let R be the root system of (G, H) with Weyl group $W = W(R)$. Fix a set Δ of simple roots for R, and let R^+ be the corresponding set of positive roots. The roots R span a real vector space $V = \mathrm{Ann}(\mathfrak{z}_G)_{\mathbb{R}} \subseteq \mathfrak{h}^*_{\mathbb{R}}$. There exists a W-invariant positive definite inner product $\langle \cdot, \cdot \rangle$ on V. Given a root α, there is an associated coroot $\alpha^\vee \in \mathfrak{h}_{\mathbb{R}}$. Using the inner product to identify V with V^*, we have $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$. We denote the Cartan integer $\alpha(\beta^\vee)$ by $n(\alpha, \beta)$. Denote by $\Delta^\vee$ the set of coroots dual to the simple roots. The *coroot lattice* Λ is the lattice inside V^* spanned by the coroots, and $\Delta^\vee$ is a basis for Λ. The fundamental group $\pi_1(H) \subseteq \mathfrak{h}$ and $(2\pi\sqrt{-1})^{-1} \cdot \pi_1(H)$ is a lattice in $\mathfrak{h}_{\mathbb{R}}$ containing Λ. From now on, we will omit the factor of $(2\pi\sqrt{-1})^{-1}$ and denote the lattice inside $\mathfrak{h}_{\mathbb{R}}$ as $\pi_1(H)$. Given $\alpha \in \Delta$, the *fundamental weight* $\varpi_\alpha \in V$ is the unique element of $\mathfrak{h}^*$ vanishing on $\mathfrak{z}_G$ and such that $\varpi_\alpha(\beta^\vee) = \delta_{\alpha\beta}$. The fundamental coweights $\varpi_\alpha^\vee \in V^*$ are defined similarly. The positive Weyl chamber $\overline{C}_0$ in $\mathfrak{h}_{\mathbb{R}}$ is defined by

$$\overline{C}_0 = \{x \in \mathfrak{h}_{\mathbb{R}} : \alpha(x) \geq 0 \text{ for all } \alpha \in \Delta\}.$$

1.2 Parabolic subgroups

The conjugacy classes of parabolic subgroups of G are in natural one-to-one correspondence with non-empty subsets $I \subseteq \Delta$. The class of parabolics associated with $I \subseteq \Delta$ has a representative P^I which is the connected subgroup of G whose Lie algebra consists of the direct sum of the Cartan subalgebra, all the positive root spaces and the negative root spaces whose roots are linear combinations of the $\alpha \in \Delta - I$. Clearly, $P^I \subseteq P^J$ if and only if $J \subseteq I$. Thus, the maximal parabolics are of the form $P^{\{\alpha\}}$ for a simple root α. We set $P^{\{\alpha\}} = P^\alpha$. By convention we define $P^\emptyset = G$.

The Levi factor of P^I is a reductive subgroup L^I. Its Lie algebra is the subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{h}$ and by the root spaces corresponding to the set of roots in the linear span of

$\Delta - I$. In particular, H is a maximal torus of L^I. Let $\Lambda_{L^I} \subseteq \Lambda$ be the lattice spanned by the coroots $\beta^\vee \in \mathfrak{h}$ dual to the simple roots $\beta \in \Delta - I$. It is a direct summand of Λ. The center $\mathfrak{z}_{L^I}$ of $\operatorname{Lie} L^I$ is given by

$$\mathfrak{z}_{L^I} = \bigcap_{\beta \in \Delta - I} \operatorname{Ker} \beta,$$

and hence $\mathfrak{z}_{L^I}$ is spanned over $\mathbb{C}$ by $\varpi_\alpha^\vee, \alpha \in I$ and $\mathfrak{z}_G$. Thus $\mathfrak{h} = \mathfrak{z}_{L^I} \oplus (\Lambda_{L^I} \otimes \mathbb{C})$. The following is the equivalent formulation in terms of root systems:

Lemma 1.2.1. *The set $\{\beta^\vee : \beta \notin I\} \cup \{\varpi_\alpha^\vee : \alpha \in I\}$ is a basis for $\Lambda \otimes \mathbb{R} = V^*$.*

Proof. Suppose that $x \in V$ and that x vanishes on all the elements in the claim. Since x vanishes on $\varpi_\alpha^\vee$ for $\alpha \in I$, it can be written as a linear combination of the $\beta \notin I$. Since the subdiagram of the Dynkin diagram for G spanned by the vertices of $\Delta - I$ is the Dynkin diagram of a semisimple group, it follows that x is zero. This shows that the given elements span V. Since the number of them is equal to the dimension of V, it follows that they are a basis. $\square$

A very similar argument shows the following:

Lemma 1.2.2. *If $x \in \mathfrak{z}_L$ is such that $\chi(x) = 0$ for all characters χ of G and $\varpi_\alpha(x) = 0$ for all $\alpha \in I$, then $x = 0$.* $\square$

A character $\chi \colon P^I \to \mathbb{C}^*$ is *dominant* if its differential $\chi_* \colon \mathfrak{h} \to \mathbb{C}$ takes nonnegative values on every simple coroot. The dominant characters of P^I which vanish on the identity component of the center of G are the nonnegative integral linear combinations of the fundamental weights $\varpi_\alpha, \alpha \in I$, which are characters of P^I. For example, if G is simply connected, the dominant characters of P^I are exactly the nonnegative integral linear combinations of the fundamental weights $\varpi_\alpha, \alpha \in I$.

2 Stratification of the space of all $(0,1)$-connections

The C^∞ G-bundle ξ_0 has a first Chern class $c = c_1(\xi_0) \in H^2(C; \pi_1(G)) = \pi_1(G) = \pi_1(H)/\Lambda$, which determines the topological type of ξ_0. Every element of $\pi_1(G)$ arises in this way. There is a slightly weaker invariant $\zeta(\xi_0)$, the image of c in $\pi_1(H)/\widehat{\Lambda}$, where $\widehat{\Lambda}$ is the saturation of Λ in $\pi_1(H)$. The quotient $\pi_1(H)/\widehat{\Lambda}$ is a lattice in $\mathfrak{h}_{\mathbb{R}}/V^* \cong (\mathfrak{z}_G)_{\mathbb{R}}$, and we shall view $\zeta(\xi_0)$ as an element of $(\mathfrak{z}_G)_{\mathbb{R}}$.

A holomorphic structure ξ on ξ_0 is determined by a $(0,1)$-connection on ξ_0. We denote by $\mathcal{A} = \mathcal{A}(\xi_0)$ the space of all $(0,1)$-connections on ξ_0. It is naturally an affine space for $\Omega^{(0,1)}(C; \operatorname{ad}\xi_0)$, and hence supports a natural structure of an infinite dimensional affine complex variety. Let $\mathcal{G}$ be the group of C^∞-automorphisms of ξ_0. Different $(0,1)$-connections determine isomorphic bundles if and only if they differ by the action of $\mathcal{G}$ on $\mathcal{A}$.

A holomorphic G-bundle ξ over C is *semistable* if $\operatorname{ad}\xi$ is a semistable vector bundle. The bundle ξ is semistable if and only if, for every irreducible representation $\pi\colon G \to GL(N)$, the associated vector bundle $\xi \times_\pi \mathbb{C}^N$ is semistable. The subset of $\mathcal{A}$ corresponding to semistable bundles is connected, open, and dense [5].

2.1 Atiyah-Bott points

Let L be a reductive subgroup of G containing H, and let $\Lambda_L \subseteq \Lambda$ be the coroot lattice of L. Thus $\pi_1(L) = \pi_1(H)/\Lambda_L$. Let $\mathfrak{z}_L \subseteq \mathfrak{h}$ be the Lie algebra of the center of L. We wish to understand the topological types of reductions of ξ_0 to an L-bundle η_0. A convenient way to record this information is through the Atiyah-Bott point of η.

Definition 2.1.1. Let L be a reductive group and let H be a maximal torus of L. Let η_0 be a C^∞ L-bundle over C, so that $c_1(\eta_0) \in \pi_1(H)/\Lambda_L$. The *Atiyah-Bott* point of η_0 is the unique point $\mu(\eta_0) \in (\mathfrak{z}_L)_\mathbb{R}$ such that, for all characters χ of L, $\chi(\mu(\eta_0)) = c_1(\eta_0 \times_\chi \mathbb{C})$. In other words, in the notation at the beginning of this section, $\mu(\eta_0) = \zeta(\eta_0)$. More generally, suppose that P is an arbitrary linear algebraic group whose unipotent radical is U and such that $P/U = L$, and that η_0 is a P-bundle. Of course, the bundle η_0 is topologically equivalent to $(\eta/U) \times_L P$ for any section of the quotient map $P \to L$. We define the Atiyah-Bott point $\mu(\eta_0)$ to be L-bundle $\mu(\eta_0/U)$.

Lemma 2.1.2. *Suppose that η_0 is a reduction of ξ_0 to a standard parabolic subgroup P^I for some $I \subseteq \Delta$, possibly empty. The Atiyah-Bott point $\mu(\eta_0)$ and the topological type of ξ_0 as a G-bundle determine the topological type of η_0/U^I as an L^I-bundle (and hence of η_0 as a P^I-bundle). Given a point $\mu \in \mathfrak{h}_\mathbb{R}$, there is a reduction of ξ to a P^I-bundle whose Atiyah-Bott point is μ if and only if the following conditions hold:*

(i) *μ lies in the Lie algebra $(\mathfrak{z}_{L^I})_\mathbb{R}$ of the center of L^I.*

(ii) *For every simple root $\alpha \in I$ we have $\varpi_\alpha(\mu) \equiv \varpi_\alpha(c) \pmod{\mathbb{Z}}$.*

(iii) *$\chi(\mu) = \chi(c)$ for all characters χ of G.*

Proof. Let $L = L^I$. Let $\widehat{\Lambda}_L$ be the saturation of Λ_L in $\pi_1(H)$, and define $\widehat{\Lambda}$ similarly. Since Λ_L is a direct summand of Λ, there is an induced injection $\widehat{\Lambda}_L/\Lambda_L \to \widehat{\Lambda}/\Lambda$. Suppose that $\alpha \in I$. Then $\varpi_\alpha\colon \mathfrak{h} \to \mathbb{C}$ vanishes on Λ_L. Of course, it takes rational values on $\pi_1(H)$ and integral values on Λ. Thus, ϖ_α determines a homomorphism $\pi_1(H)/\Lambda$ to $\mathbb{Q}/\mathbb{Z}$, and thus by restriction a homomorphism $\widehat{\Lambda}/\Lambda \to \mathbb{Q}/\mathbb{Z}$. In fact, an easy argument shows that the sequence

$$0 \to \widehat{\Lambda}_L/\Lambda_L \to \widehat{\Lambda}/\Lambda \to \bigoplus_{\alpha\in I} \mathbb{Q}/\mathbb{Z}$$

is exact, where the map on the right is the one induced by $\oplus_{\alpha\in I}\varpi_\alpha$. The subspace $\Lambda_L \otimes \mathbb{R} \subseteq \mathfrak{h}_\mathbb{R}$ is a complementary subspace in $\mathfrak{h}_\mathbb{R}$ to $(\mathfrak{z}_L)_\mathbb{R}$. Thus projection from $\mathfrak{h}_\mathbb{R}$ to $(\mathfrak{z}_L)_\mathbb{R}$ defines

a homomorphism from $\pi_1(H)/\Lambda_L$ to $(\mathfrak{z}_L)_\mathbb{R}$, whose kernel is $\widehat{\Lambda}_L/\Lambda_L$. Summarizing, we have a commutative diagram with exact columns and exact first row:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widehat{\Lambda}_L/\Lambda_L \arrow[r] \arrow[d] & \widehat{\Lambda}/\Lambda \arrow[r] \arrow[d] & \bigoplus_{\alpha\in I}\mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\
 & \pi_1(H)/\Lambda_L \arrow[r] \arrow[d] & \pi_1(H)/\Lambda \arrow[r] \arrow[d] & \bigoplus_{\alpha\in I}\mathbb{Q}/\mathbb{Z} \\
 & \pi_1(H)/\widehat{\Lambda}_L \arrow[r] & \pi_1(H)/\widehat{\Lambda} &
\end{tikzcd}$$

The point $\mu(\eta_0)$ is the image of $c_1(\eta_0)$ under the homomorphism $\pi_1(H)/\Lambda_L \to (\mathfrak{z}_L)_\mathbb{R}$. It follows that $\mu(\eta_0)$ determines $c_1(\eta_0)$ up to an element of $\widehat{\Lambda}_L/\Lambda$. Thus, two distinct points $\gamma \neq \gamma' \in \pi_1(H)/\Lambda_L$ with the same image in $\mathfrak{h}_\mathbb{R}/(\Lambda_L\otimes\mathbb{R})$ have distinct images in $\pi_1(H)/\Lambda$. This shows that given μ, there is at most one lift of μ to a point $\gamma \in \pi_1(H)/\Lambda_L$ whose image in $\pi_1(H)/\Lambda$ is c, and hence that $\mu(\eta_0)$ and c (or equivalently the topological type of ξ_0) determine the topological type of η_0/U as an L-bundle. Of course, the topological type of η_0/U as an L-bundle determines the topological type of η_0 as a P-bundle.

Next let us show that the Atiyah-Bott point $\mu(\eta_0)$ satisfies the three conditions stated in the lemma. By construction it lies in the Lie algebra of the center of L. Clearly, since $c_1(\eta_0) \in \pi_1(H)/\Lambda_L$ maps to c in $\pi_1(H)/\Lambda$, the congruence given in the second item holds. Lastly, since $\chi(\mu(\eta_0)) = \chi(c_1(\eta_0))$ for every character χ of L, and since under the inclusion of $L \subseteq G$ the element $c_1(\eta_0)$ maps to c, it follows that $\chi(\mu(\eta_0)) = \chi(c)$ for every character of G.

Conversely, fix a point μ satisfying the three conditions above. We shall show that μ is the Atiyah-Bott point of a reduction to P^I of the C^∞ bundle ξ_0. We claim that there is a C^∞ L-bundle η_0 such that $c_1(\eta_0)$ projects to μ and such that $c_1(\eta_0 \times_L G) = c$. It suffices to show that there is an element $c_L \in \pi_1(H)/\Lambda_L$ which projects to μ and whose image in $\pi_1(H)/\Lambda$ is c, for then there is a C^∞ L-bundle η_0 with $c_1(\eta) = c_L$, and this bundle has the required properties. Choose $\widetilde{c} \in \pi_1(H)$ lifting c. For all $\alpha \in I$, $\varpi_\alpha(\mu - \widetilde{c}) \in \mathbb{Z}$. Thus there exists a $\lambda \in \Lambda$ such that $\varpi_\alpha(\mu) = \varpi_\alpha(\lambda + \widetilde{c})$ for all $\alpha \in I$. Let c_L be the image of $\lambda + \widetilde{c}$ in $\pi_1(H)/\Lambda_L$. Then c_L maps to $c \in \pi_1(H)/\Lambda$, and the image μ' of c_L satisfies: $\varpi_\alpha(\mu) = \varpi_\alpha(\mu')$ for all $\alpha \in I$. Clearly, for every character χ of G, $\chi(\mu) = \chi(\mu')$. By Lemma 1.2.2, $\mu = \mu'$, and so c_L is the required element. This completes the proof of Lemma 2.1.2. $\square$

Definition 2.1.3. A pair (μ, I) consisting of a point $\mu \in \mathfrak{h}_\mathbb{R}$ and a subset $I \subseteq \Delta$ is said to be *of Atiyah-Bott type for* c (or for ξ_0) if and only if the three conditions given in Lemma 2.1.2 are satisfied. If (μ, I) and (μ, I') are Atiyah-Bott pairs of type c, then so is $(\mu, I \cap I')$. Thus, one can always choose I to be minimal so that the conditions hold. This minimal I consists of all $\alpha \in \Delta$ for which $\alpha(\mu) \neq 0$. A point $\mu \in \mathfrak{h}_\mathbb{R}$ is said to be *of*

Atiyah-Bott type for c if there is a subset $I \subseteq \Delta$ such that (μ, I) is a pair of Atiyah-Bott type for c.

Corollary 2.1.4. *Given a pair (μ, I) of Atiyah-Bott type for c, there is a reduction of ξ_0 to an L^I-bundle η_0 whose Atiyah-Bott point is μ, and η_0 is unique up to C^∞ isomorphism.* □

2.2 The Harder-Narasimhan reduction

Recall that every unstable holomorphic G-bundle ξ has a canonical reduction of its structure group to a conjugacy class of parabolic subgroups. We can always choose a standard parabolic P^I in this class. We call this reduction the *Harder-Narasimhan reduction* and P the *Harder-Narasimhan parabolic* of ξ. If ξ is semistable, then by convention we take as the Harder-Narasimhan reduction the trivial reduction to the group $G = P^\emptyset$. The following summarizes the basic properties of this reduction [5], [1], [3, Corollary 2.11].

Lemma 2.2.1. *Let ξ be a holomorphic G-bundle over C. A reduction of ξ to a P^I-bundle η is the Harder-Narasimhan reduction if and only if*

(i) *η/U^I is a semistable L^I-bundle.*

(ii) *$\mu(\eta) \in \overline{C}_0$.*

(iii) *If $\alpha \in I$, then $\alpha(\mu(\eta)) > 0$.* □

Corollary 2.2.2. *Let $\mu \in \mathfrak{h}_\mathbb{R}$ be of Atiyah-Bott type for c and let $I = \{\alpha \in \Delta : \alpha(\mu) > 0\}$. Then there is a holomorphic G-bundle structure on ξ_0 which has Harder-Narasimhan reduction to a subgroup P^I with Atiyah-Bott point μ if and only if $\mu \in \overline{C}_0$.*

Proof. Since every C^∞ bundle has a semistable holomorphic structure, this is clear from the previous lemma and Lemma 2.1.2. □

2.3 The strata

Following Shatz and Atiyah-Bott we define a stratification of the space of $(0,1)$-connections on ξ_0.

Definition 2.3.1. Fix a point $\mu \in \overline{C}_0$ of Atiyah-Bott type for c. The *stratum* $\mathcal{C}_\mu \subseteq \mathcal{A}$ is the set of all $(0,1)$-connections defining holomorphic structures on ξ_0 whose Harder-Narasimhan reduction has Atiyah-Bott point equal to μ.

In particular, if $c = \zeta + v$ with $\zeta \in \mathfrak{z}_G$ and $v \in V^*$, then ζ is of Atiyah-Bott type for c and $\mathcal{C}_\zeta$ is exactly the open dense set of $(0,1)$-connections defining semistable G-bundles. Thus for example if G is semisimple, then the stratum of semistable bundles is $\mathcal{C}_0$.

The strata are preserved by the action of $\mathcal{G}$. The union of these strata over all $\mu \in \overline{C}_0$ of Atiyah-Bott type for ξ_0 is $\mathcal{A}$.

We say that a holomorphic bundle structure on ξ_0, or equivalently a holomorphic bundle whose underlying topological bundle is isomorphic to ξ_0, is contained in the stratum $\mathcal{C}_\mu$ if all $(0,1)$-connections determining holomorphic bundles isomorphic to it are in this stratum.

Proposition 2.3.2. *Suppose that $\mu \in \overline{C}_0$ is a point of Atiyah-Bott type for c. Let $a, b \in \mathcal{C}_\mu$. Then there is a connected complex space S, a holomorphic family of $(0,1)$-connections $A_s, s \in S$, and points $s_1, s_2 \in S$ such that $A_s \in \mathcal{C}_\mu$ for all $s \in S$ and A_{s_1} is gauge equivalent to a and A_{s_2} is gauge equivalent to b. In particular, $\mathcal{C}_\mu/\mathcal{G}$ is connected.*

Proof. Lemma 2.1.2 shows that the parabolic subgroup and the topological type of the L-bundle η_0 are determined by μ and the topological type of ξ_0. Given two connections a_L, b_L on η_0 which define semistable holomorphic structures, an open dense set of the line in $\mathcal{A}(\eta_0)$ joining them will also define semistable holomorphic structures. We can use this line to join the corresponding $(0,1)$-connections a' and b' on ξ_0. The space of holomorphic P-bundles with a given L-reduction is an affine space, by [4, Appendix]. After choosing a C^∞ trivialization of these spaces, we can find holomorphic, connected families of $(0,1)$-connections joining a' to a and b' to b. $\square$

The strata are locally closed in the Zariski topology in the following sense. If T is a finite dimensional parameter space for an algebraic family of holomorphic structures on ξ and μ is a point of Atiyah-Bott type for ξ, then the subspace of T consisting of points parametrizing bundles contained in $\mathcal{C}_\mu$ is a locally closed subspace of T with respect to the Zariski topology. An analogous statement for the classical topology holds for analytic families of holomorphic structures on ξ.

Definition 2.3.3. Following Atiyah-Bott, we say that $\mathcal{C}_{\mu_1} \preceq_r \mathcal{C}_{\mu_2}$ if there exists a holomorphic G-bundle ξ in $\mathcal{C}_{\mu_2}$ and an arbitrarily small deformation of it to a bundle in $\mathcal{C}_{\mu_1}$. In light of the above remarks this is equivalent to the existence of a holomorphic family of G-bundles Ξ over $C \times T$, where T is connected, and a point $t_0 \in T$, such that $\Xi_t \in \mathcal{C}_{\mu_1}$ for $t \neq t_0$, and such that $\Xi_{t_0} \in \mathcal{C}_{\mu_2}$. The relation $\preceq_r$ generates a transitive relation which we denote by $\preceq$.

2.4 The Atiyah-Bott ordering

Definition 2.4.1. We define the *Atiyah-Bott partial ordering* on $\mathfrak{h}_\mathbb{R}$ as follows. Given $x \in \mathfrak{h}_\mathbb{R}$, let $\widehat{W \cdot x}$ denote the convex hull of the finite set $W \cdot x$. For $x, y \in \mathfrak{h}_\mathbb{R}$, we define $x \geq y$ if and only if $y \in \widehat{W \cdot x}$. Notice that $x \geq y$ if and only if the projections of x and y into $(\mathfrak{z}_G)_\mathbb{R}$ are equal and $v^*(x) \geq v^*(y)$ where $v^*(x)$ and $v^*(y)$ are the projections of x and y into V^*.

For points in $\overline{C}_0$, there is a simple characterization of the ordering.

Lemma 2.4.2. *Suppose that $x, y \in \overline{C}_0$. Then $x \geq y$ if and only if:*

(i) *For every simple root α, $\varpi_\alpha(x) \geq \varpi_\alpha(y)$.*

(ii) *The projections of x and y into $\mathfrak{z}_G$ are equal.*

Proof. Since the Weyl group acts trivially on $\mathfrak{z}_G$, it suffices to divide out by $\mathfrak{z}_G$. We may thus assume that G is semisimple. By [1, Lemma 12.14], for $x, y \in \overline{C}_0$, $x \geq y$ if and only if $\langle t, x\rangle \geq \langle t, y\rangle$ for all $t \in \overline{C}_0$. But the simplicial cone $\overline{C}_0$ is spanned over $\mathbb{R}^+$ by the elements $\varpi_\alpha^\vee$, $\alpha \in \Delta$, and $\langle \varpi_\alpha^\vee, x\rangle = c\varpi_\alpha(x)$ for some positive constant c. Thus $x \geq y$ if and only if $\varpi_\alpha(x) \geq \varpi_\alpha(y)$ for every $\alpha \in \Delta$. □

The next corollary says that on $\overline{C}_0$ the Atiyah-Bott partial ordering really is a partial ordering:

Corollary 2.4.3. *Suppose that $x_1, x_2 \in \overline{C}_0$ are such that $x_1 \leq x_2$ and $x_2 \leq x_1$. Then $x_1 = x_2$. More generally, given $x_1, x_2 \in \mathfrak{h}_\mathbb{R}$, $x_1 \leq x_2 \leq x_1$ if and ony if x_1 and x_2 are conjugate under W.* □

The relevance of the Atiyah-Bott ordering to the ordering of strata in the space of $(0,1)$-connections is given by the following theorem [1]:

Theorem 2.4.4. (Atiyah-Bott) *Suppose that $\mathcal{C}_{\mu_1} \preceq_r \mathcal{C}_{\mu_2}$, i.e., suppose that there is a bundle in the stratum $\mathcal{C}_{\mu_2}$ and an arbitrarily small deformation of it which is a bundle in the stratum $\mathcal{C}_{\mu_1}$. Then $\mu_1 \leq \mu_2$.* □

Corollary 2.4.5. *If $\mathcal{C}_{\mu_1} \preceq \mathcal{C}_{\mu_2}$, then $\mu_1 \leq \mu_2$ in the Atiyah-Bott partial ordering. In particular, we see that the transitive relation $\preceq$ is a partial ordering on the set of strata $\mathcal{C}_\mu$.*

The rest of this paper is devoted to establishing a converse to this result. For the case of curves of genus one, we have the following strong converse:

Theorem 2.4.6. *Suppose that $g(C) = 1$, and that $\mu_1 \leq \mu_2$ in the Atiyah-Bott ordering and that ξ is a holomorphic bundle in $\mathcal{C}_{\mu_2}$. Then there is an arbitrarily small deformation ξ' of ξ contained in $\mathcal{C}_{\mu_1}$.*

For higher genus, there is a weaker version of Theorem 2.4.6:

Theorem 2.4.7. *Let C be a smooth curve of genus at least one. Then $\mathcal{C}_{\mu_1} \preceq \mathcal{C}_{\mu_2}$, if and only if $\mu_1 \leq \mu_2$.*

It follows from Theorem 2.4.6 that in the case of curves of genus one, the stratification is in fact a stratification in a strong sense as the next corollary shows.

Corollary 2.4.8. *Suppose that $g(C) = 1$. Let $\mu \in \overline{C}_0$ be a point of Atiyah-Bott type for c. The closure of the stratum $\mathcal{C}_\mu$ in $\mathcal{A}$ is equal to $\bigcup_{\mu' \geq \mu} \mathcal{C}_{\mu'}$ where μ' ranges over points in $\overline{C}_0$ of Atiyah-Bott type for c.*

Proof. Suppose that $\mu' \in \overline{C}_0$ is a point of Atiyah-Bott type with $\mu' \geq \mu$. Let A be a $(0,1)$-connection in $\mathcal{C}_{\mu'}$. Let η' be the holomorphic G-bundle determined by A. According to Theorem 2.4.6 there is an arbitrarily small deformation of η' to a holomorphic G-bundle η contained in $\mathcal{C}_\mu$. We can extend A to a $(0,1)$-connection on this deformation. A C^∞ trivialization of the deformation allows us to view all the $(0,1)$-connections in the family as connections on ξ_0. The resulting $(0,1)$-connection determining η is then arbitrarily close to A in the space of $(0,1)$-connections. Thus, every neighborhood of A in $\mathcal{A}$ contains a $(0,1)$-connection in $\mathcal{C}_\mu$. Hence, A is contained in the closure of $\mathcal{C}_\mu$.

This proves that the subspace $\bigcup_{\mu' \geq \mu} \mathcal{C}_{\mu'}$, where μ' ranges over points in $\overline{C}_0$ of Atiyah-Bott type for c, is contained in the closure of $\mathcal{C}_\mu$. The Atiyah-Bott result is that the closure of $\mathcal{C}_\mu$ is contained in this union. Hence, the closure of $\mathcal{C}_\mu$ is equal to this union. □

3 Harmonic functions on a Dynkin diagram

The Cartan matrix $\big(n(\alpha,\beta)\big)_{\alpha,\beta\in\Delta}$ has two fundamental properties: Its off-diagonal entries are non-positive, and its inverse is a positive matrix. The fact that these hold for Δ and for all non-empty subsets of Δ allows us to establish a theory of harmonic functions on $\Delta^\vee$ with results paralleling those for harmonic functions on a compact manifold.

3.1 The basic definitions

A point $x \in V^*$ can be written uniquely

$$x = \sum_{\alpha\in\Delta} r_\alpha \alpha^\vee$$

with $r_\alpha \in \mathbb{R}$. Thus, we can view x as a function f_x on $\Delta^\vee$ by $f_x(\alpha^\vee) = r_\alpha = \varpi_\alpha(x)$. This determines a linear isomorphism between V^* and the space of functions $\Delta^\vee \to \mathbb{R}$.

Definition 3.1.1. Let $f\colon \Delta^\vee \to \mathbb{R}$ be a function. It is *harmonic at* $\alpha^\vee \in \Delta^\vee$ if

$$\sum_{\beta\in\Delta} n(\alpha,\beta) f(\beta^\vee) = 0.$$

The function f is *superharmonic at* $\alpha^\vee$ if

$$\sum_{\beta\in\Delta} n(\alpha,\beta) f(\beta^\vee) \geq 0,$$

and *subharmonic at* $\alpha^\vee$ if

$$\sum_{\beta \in \Delta} n(\alpha,\beta) f(\beta^\vee) \leq 0.$$

Thus, for $x \in V^*$, f_x is superharmonic, harmonic, or subharmonic at α according to whether $\alpha(x) \geq 0$, $\alpha(x) = 0$, or $\alpha(x) \leq 0$. Let $A \subseteq \Delta^\vee$. Then f is *harmonic except at* A if it is harmonic at every $\alpha^\vee \in \Delta^\vee - A$. It is *harmonic* if it is harmonic at every $\alpha^\vee \in \Delta^\vee$. Similarly, one defines the notions of superharmonic and subharmonic except at A and superharmonic and subharmonic.

There is a more geometric way of viewing the harmonic condition at $\alpha \in \Delta^\vee$. It is a local condition just involving the values of f on the unit disk centered at $\alpha^\vee$ – an average value condition where the values are weighted by the Cartan integers. For $\alpha^\vee \in \Delta^\vee$, let $\mathbf{s}(\alpha^\vee) = \{\beta^\vee \in \Delta^\vee : n(\alpha,\beta) < 0\}$ be the "sphere of radius one" around $\alpha^\vee$. Since the only positive Cartan integers are the diagonal ones which are equal to 2, for a function $f\colon \Delta^\vee \to \mathbb{R}$ it is immediate from the definition that f is superharmonic at $\alpha^\vee$ if and only if

$$2f(\alpha^\vee) \geq - \sum_{\beta^\vee \in \mathbf{s}(\alpha)} n(\alpha,\beta) f(\beta^\vee).$$

There are similar descriptions of when f is harmonic or subharmonic at $\alpha^\vee$.

3.2 Basic properties of harmonic functions on $\Delta^\vee$

Definition 3.2.1. If f and g are functions from $\Delta^\vee$ to $\mathbb{R}$ we say that $f \geq g$ if $f(\alpha^\vee) \geq g(\alpha^\vee)$ for all $\alpha^\vee \in \Delta^\vee$.

Lemma 3.2.2. *For $x \in V^*$, the corresponding function f_x is superharmonic if and only if $x \in \overline{C}_0$.*

Proof. As we have seen, f_x is superharmonic at $\alpha^\vee$ if and only if $\alpha(x) \geq 0$. Consequently, f_x is superharmonic if and only if $x \in \overline{C}_0$. □

Lemma 3.2.3. *A function $f\colon \Delta^\vee \to \mathbb{R}$ is superharmonic if and only if it is a nonnegative linear combination of $\{f_{\varpi_\alpha^\vee}\}_{\alpha\in\Delta}$. A superharmonic function f on $\Delta^\vee$ satisfies $f \geq 0$. If f is harmonic, then it is zero.*

Proof. Since the simplicial cone $\overline{C}_0$ is spanned by $\{\varpi_\alpha^\vee\}_{\alpha\in\Delta}$, the first statement is clear. Since $\varpi_\beta(\varpi_\alpha^\vee) \geq 0$ for all $\alpha, \beta \in \Delta$, by e.g. [2, p. 168], it follows that, if f is superharmonic, then $f \geq 0$. Finally, if f is harmonic, then f and $-f$ are both superharmonic, so that $f = 0$. □

Lemma 3.2.4. *Let $A \subseteq \Delta^\vee$, and let f, g, and h be functions from $\Delta^\vee$ to $\mathbb{R}$ with f harmonic except at A, g superharmonic except at A, and h subharmonic except at A. Suppose that $g(\alpha^\vee) \geq f(\alpha^\vee) \geq h(\alpha^\vee)$ for all $\alpha^\vee \in A$. Then $g \geq f \geq h$.*

Proof. Since the negative of a harmonic function is harmonic and the negative of a superharmonic function is subharmonic, the result for a subharmonic function h with $f \geq h$ follows from that for a superharmonic function g with $g \geq f$. After replacing g by $g - f$, we may assume that g is superharmonic except at A and that $g|A \geq 0$, and wish to show that $g \geq 0$. Suppose that $g = f_x$ for $x \in V^*$ with $x = \sum_{\beta^\vee \in \Delta^\vee} x_\beta \beta^\vee$, where the x_β are given nonnegative real numbers for $\beta^\vee \in A$. The function g is superharmonic except at A if and only if $\alpha(x) \geq 0$ for all $\alpha^\vee \notin A$ if and only

$$\sum_{\gamma^\vee \in \Delta^\vee - A} x_\gamma n(\alpha, \gamma) \geq - \sum_{\beta^\vee \in A} x_\beta n(\alpha, \beta) = v_\alpha,$$

say, where the v_α are given nonnegative real numbers since $n(\alpha, \beta) \leq 0$. It suffices to show that $x_\gamma \geq 0$ for all $\gamma^\vee \notin A$. This follows since the inverse of the Cartan matrix for the subdiagram corresponding to $\Delta^\vee - A$ has nonnegative entries. $\square$

Lemma 3.2.5. *Fix $A \subseteq \Delta$. Given a function $f_0\colon A \to \mathbb{R}$, there is a unique extension $f\colon \Delta^\vee \to \mathbb{R}$ of f_0 which is harmonic except at A. If $f_0 \geq 0$, then $f \geq 0$.*

Proof. By Lemma 1.2.1 appplied to the dual root system, the set $\{\varpi_\beta\}_{\beta \in A} \cup (\Delta - A)$ is a basis for V. Thus if $f = f_x$ for $x \in V^*$, then x is uniquely determined by the conditions $\alpha(x) = 0$ for $\alpha \notin A$ and $\varpi_\beta(x) = f(\beta^\vee)$ for $\beta \in A$. The positivity statement is the special case of the previous lemma, applied to the inequality $f \geq 0$ on A and viewing f as superharmonic except at A and 0 as harmonic. $\square$

Lemma 3.2.6. *If $\mu, \mu' \in \overline{C}_0$, then $\mu \geq \mu'$ in the Atiyah-Bott ordering if and only if $f_\mu \geq f_{\mu'}$. If $\mu, \mu' \in -\overline{C}_0$, then $\mu \geq \mu'$ in the Atiyah-Bott ordering if and only if $f_{\mu'} \geq f_\mu$.*

Proof. The first statement is immediate from Lemma 2.4.2. Suppose $\mu, \mu' \in -\overline{C}_0$. Then $\mu \geq \mu'$ in the Atiyah-Bott ordering if and only if $\mu' \in \widehat{W \cdot \mu}$, if and only if $-\mu' \in \widehat{W \cdot (-\mu)}$. Since $-\mu, -\mu' \in \overline{C}_0$, it follows from the first statement that this holds if and only if $f_{-\mu} \geq f_{-\mu'}$ which is clearly equivalent to the statement $f_\mu \leq f_{\mu'}$. $\square$

3.3 Examples

Let us suppose that $G = SL(n)$. Then we view $\Delta^\vee$ as the points $\{1, \ldots, n-1\}$ in the interval $I = [0, n]$ with

$$|a - b| = 1 \text{ if and only if } n(a, b) = -1 \text{ for all } a, b \in \Delta^\vee. \tag{1}$$

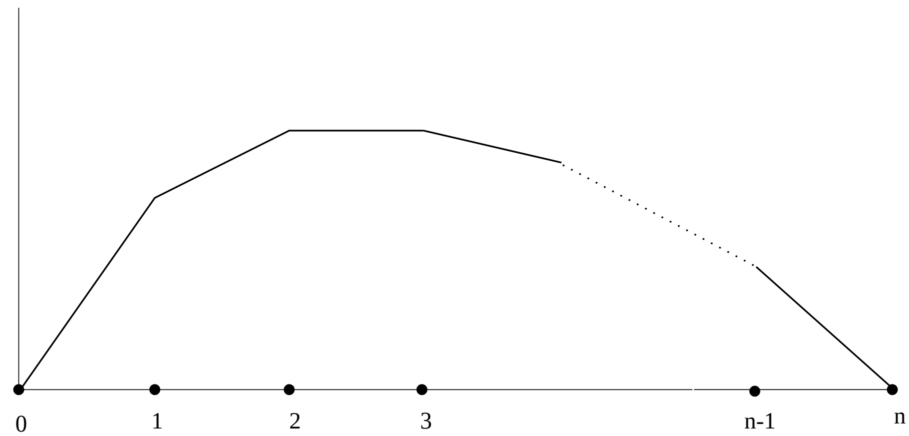

Figure 1: The Graph of a Superharmonic Function for A_{n-1}

Given a function $f\colon \Delta^\vee \to \mathbb{R}$ we extend it to a continuous function $\hat{f}\colon I \to \mathbb{R}$ by first requiring that $\hat{f}(0) = \hat{f}(n) = 0$ and that $\hat{f}$ is linear on each inteval $[k, k+1]$, $k = 0, \ldots, n-1$. Then f is superharmonic if and only if $\hat{f}$ is a convex function on I. Furthermore, f is harmonic at $a \in \Delta^\vee$ if and only if $\hat{f}$ is linear at a. (See Figure 1.)

As Atiyah-Bott point out, $f \geq g$ in their ordering if and only if the graph of $\hat{f}$ lies above that of $\hat{g}$.

Now suppose that G is a simple group of type D_n or E_6, E_7, or E_8, so that the Dynkin diagram for $\Delta^\vee$ is a union of three simple chains (Dynkin diagrams of A-type) $\Delta_1^\vee, \Delta_2^\vee, \Delta_3^\vee$ meeting at the trivalent vertex a of $\Delta^\vee$. For $i = 1, 2, 3$ let $\ell_i = \#\Delta_i^\vee$ and let I_i be the interval $[0, \ell_i]$. Identify $\Delta_i^\vee$ with $\{1, 2, \ldots, \ell_i\} \subset I_i$ satisfying Condition 1 in such a way that the trivalent vertex a is identified with $\ell_i \in I_i$. Let $T = \bigcup_{i=1}^3 I_i$ where $I_i \cap I_j = \{a\}$ for all $i \neq j$. There is a unique embedding of $\Delta^\vee \subset T$ consistent with the given embeddings of $\Delta_i^\vee \subset I_i$. Given a function $f\colon \Delta^\vee \to \mathbb{R}$ we extend it to a continuous function $\hat{f}\colon T \to \mathbb{R}$ by requiring that $\hat{f}$ vanishes on $0 \in I_i$, for $i = 1, 2, 3$ and by requiring that $\hat{f}$ is linear on each interval of the form $[k, k+1] \subset I_i$, $k = 0, 1, \ldots, \ell_i - 1$. Let $s_i = \hat{f}(a) - \hat{f}(\ell_i - 1)$ be the "slope" at a. The function f is superharmonic if and only if (i) $\hat{f}|I_i$ is convex for each $i = 1, 2, 3$, and (ii) $\sum_{i=1}^3 s_i \geq \hat{f}(a)$. The function f is harmonic at $b \in \Delta_i^\vee - \{a\}$ if and only if $\hat{f}|I_i$ is linear at b. The function f is harmonic at a if and only if the inequality in (ii) above is an equality. (See Figure 2.)

Lastly, suppose that G is a simple group of type B_n, C_n, F_4 or G_2 so that the Dynkin diagram of G is a chain with a single multiple bond. Identify $\Delta^\vee$ with the points $\{1, \ldots, n\}$ in the inteval $I = [0, n+1]$ satisfying Condition 1. Suppose that the multiple bond in the Dynkin diagram connects ℓ and $\ell + 1$ with ℓ identified with a short coroot a (which then is then the special vertex) and $\ell + 1$ being identified with a long coroot in $\Delta^\vee$. Let $m \geq 2$ be the multiplicity of the bond. Let $f\colon \Delta^\vee \to \mathbb{R}$ be a function. We extend f to a function $\hat{f}\colon I \to \mathbb{R}$ by requiring that $\hat{f}(0) = \hat{f}(n+1) = 0$ and that $\hat{f}$ be linear on each inteval of the form $[k, k+1]$, $k = 0, \ldots, n$. Let $s_1 = \hat{f}(\ell) - \hat{f}(\ell - 1)$ and let $s_2 = \hat{f}(\ell) - \hat{f}(\ell + 1)$.

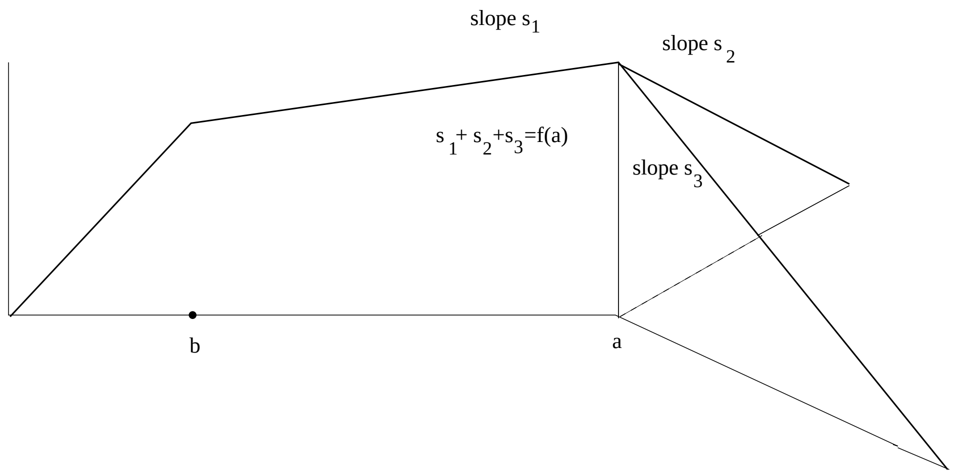

Figure 2: The Graph of a Function Superharmonic at b and Harmonic Elsewhere

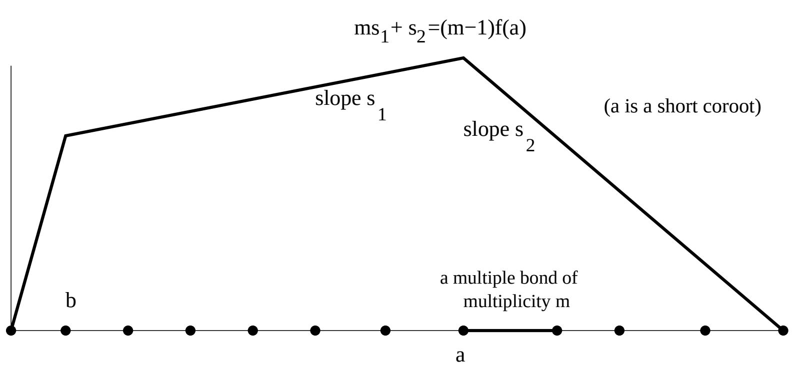

Figure 3: The Graph of a Function Superharmonic at b and Harmonic Elsewhere

Then f is superharmonic if and only if (i) $\hat{f}|[0,\ell]$ and $\hat{f}|[\ell, n+1]$ are convex and (ii) $(m-1)f(a) \leq ms_1 + s_2$. The function f is harmonic at $b \in \Delta^\vee - \{a\}$ if and only if $\hat{f}$ is linear at b, and f is harmonic at a if and only if the inequality in (ii) above is an equality. (See Figure 3.)

3.4 Relation to points of Atiyah-Bott type

We now link the above discussion with points of Atiyah-Bott type. Recall that the strata $\mathcal{C}_\mu$ of $\mathcal{A} = \mathcal{A}(\xi)$ are indexed by points μ of Atiyah-Bott type for c and such that $\mu \in \overline{C}_0$.

Proposition 3.4.1. *Write $c = \zeta + v$ with $\zeta \in (\mathfrak{z}_G)_\mathbb{R}$ and $v \in V^*$. Let $\mu \in \mathfrak{h}_\mathbb{R}$. Decompose $\mu = \zeta' + \nu$, with $\zeta' \in (\mathfrak{z}_G)_\mathbb{R}$ and $\nu \in V^*$. Let $I \subseteq \Delta$. Then (μ, I) is of Atiyah-Bott type for c if and only if*

(i) *The function f_ν is harmonic except at I;*

(ii) *For each $\alpha^\vee \in I$, $f_\nu(\alpha^\vee) \equiv \varpi_\alpha(c) \pmod{\mathbb{Z}}$.*

(iii) *$\zeta' = \zeta$.*

Moreover, $\mu \in \overline{C}_0$ if and only if f_ν is superharmonic.

Proof. The condition $\zeta' = \zeta$ is equivalent to the condition that for each character χ of G we have $\chi(\mu) = \chi(c)$. The condition that f_ν is harmonic at $\alpha^\vee$ means that $\alpha(\mu) = 0$. The condition that $f_\nu(\alpha^\vee) \equiv \varpi_\alpha(c) \pmod{\mathbb{Z}}$ is equivalent to $\varpi_\alpha(\mu) \equiv \varpi_\alpha(c) \pmod{\mathbb{Z}}$. Thus, by Lemma 2.1.2, (μ, I) is of Atiyah-Bott type for c. The final statement follows from Lemma 3.2.2. ∎

4 Proof of the main theorems

In this section we prove the two main theorems of this paper. In the first subsection, we describe a combinatorial problem which involves moving from one Atiyah-Bott point of type c in $\overline{C}_0$ to a smaller one. In the second, we relate this combinatorial problem to operations on bundles. Taken together, these results give a proof of Theorem 2.4.6. We then describe the modifications necessary to prove the weaker result Theorem 2.4.7 in the case of higher genus.

4.1 A combinatorial discussion

Let $\mu \in \mathfrak{h}_\mathbb{R}$. Then we define the function $f_\mu \colon \Delta^\vee \to \mathbb{R}$ as follows. We decompose $\mu = \zeta' + \nu$ with $\zeta' \in (\mathfrak{z}_G)_\mathbb{R}$ and $\nu \in V^*$ and we set $f_\mu = f_\nu$. Note that $\mu \in \overline{C}_0$ if and only if f_μ is superharmonic. For points $\mu, \mu' \in \overline{C}_0$ we have $f_\mu \geq f_{\mu'}$ if and only if $\mu \geq \mu'$ in the ordering given in Definition 2.4.1. Given $f_\mu \geq f_{\mu'}$, the following theorem gives a description of a

sequence of moves which begins with μ and ends with μ'. Although we shall primarily be interested in the case where μ and μ' lie in $\overline{C}_0$, we cannot assume that the intervening points μ_i lie in $\overline{C}_0$ and shall not make any assumptions on μ and μ'.

Theorem 4.1.1. *Suppose that we have two pairs (μ, I) and (μ', I') of Atiyah-Bott type for c with $f_\mu \geq f_{\mu'}$. Then there is a sequence of pairs of Atiyah-Bott type for c, $(\mu, I) = (\mu_0, I_0), \ldots, (\mu_n, I_n) = (\mu', I')$ with $f_{\mu_i} \geq f_{\mu_{i+1}}$ for $0 \leq i \leq n-1$, and such that, for each i, $0 \leq i \leq n-1$, (μ_{i+1}, I_{i+1}) is obtained from (μ_i, I_i) by one of the following moves.*

(1) *$I_{i+1} = I_i \cup \{a\}$, $f_{\mu_i}|_{I_i} = f_{\mu_{i+1}}|_{I_i}$, and $f_{\mu_{i+1}}$ is subharmonic at a. Furthermore, if $\nu \in \mathfrak{h}_{\mathbb{R}}$ is a point of Atiyah-Bott type for c and $f_{\mu_i} \geq f_\nu \geq f_{\mu_{i+1}}$ then either $f_\nu = f_{\mu_i}$ or $f_\nu = f_{\mu_{i+1}}$.*

(2) *$I_{i+1} \subseteq I_i$, $f_{\mu_i}|_{I_{i+1}} = f_{\mu_{i+1}}|_{I_{i+1}}$.*

(3) *$I_i = I_{i+1}$ and there is $a \in I_i$ such that the restrictions of f_{μ_i} and $f_{\mu_{i+1}}$ agree on $I_i - \{a\}$.*

Proof. There are only finitely many points μ_1 of Atiyah-Bott type for c with $\mu \geq \mu_1 \geq \mu'$. Thus, it suffices to show that if $f_\mu > f_{\mu'}$, and if there is no point μ'' of Atiyah-Bott type for c with $f_\mu > f_{\mu''} > f_{\mu'}$, then a finite sequence of the moves listed can be applied to (μ, I), each one not increasing μ, to produce (μ', I'). First suppose that $I' \not\subseteq I$. Then choose $b_0 \in I' - I$ and define a function f_0 which is harmonic on $\Delta^\vee - (I \cup \{b_0\})$ with $f_0(a) = f_\mu(a)$ for all $a \in I$ and $f_0(b_0) = f_{\mu'}(b_0)$. By Lemma 3.2.5 there is a unique such function. Clearly, by Proposition 3.4.1, since μ and μ' are of Atiyah-Bott type for c, the resulting function f_0 is determined by a point ν_0 in $\mathfrak{h}_{\mathbb{R}}$ such that $(\nu_0, I \cup \{b_0\})$ of Atiyah-Bott type for c.

Since $f_0(b_0) \leq f_\mu(b_0)$, it also follows from Lemma 3.2.5 that $f_\mu \geq f_0$. Since μ is harmonic at b_0, it follows easily from the definitions, Lemma 3.2.4, and the fact that $f_\mu \geq f_0$ that f_0 is subharmonic at b_0. Suppose that $f_0 \not\geq f_{\mu'}$. Then there exists a $b \in \Delta^\vee$ such that $f_0(b) < f_{\mu'}(b)$. Since f_0 and $f_{\mu'}$ are both harmonic except at $I \cup I'$, by Lemma 3.2.5 there exists a $b_1 \in I \cup I'$ with $f_0(b_1) < f_{\mu'}(b_1)$. Since $f_{\mu'}|I \cup \{b_0\} \leq f_0|I \cup \{b_0\}$, it follows that $b_1 \in I' - (I \cup \{b_0\})$. Now perform the same construction with b_1 replacing b_0, producing a function $f_1 \leq f_\mu$ which is harmonic except at $I \cup \{b_1\}$ and subharmonic at b_1. Let $\nu_1 \in \mathfrak{h}$ be the point of Atiyah-Bott type for c for which $f_{\nu_1} = f_1$. Since $f_1|I \cup \{b_1\} > f_0|I \cup \{b_1\}$ and since f_1 is subharmonic except at I and harmonic except at $I \cup \{b_1\}$, it follows from Lemma 3.2.4 that $f_\mu \geq f_1 > f_0$. Continuing in this way, we can eventually choose $b \in I' - I$ and a corresponding point $\nu \in \mathfrak{h}_{\mathbb{R}}$ with $(\nu, I \cup \{b\})$ of Atiyah-Bott type for c such that the corresponding function f_ν has the property that $f_\mu \geq f_\nu \geq f_{\mu'}$. Clearly with this choice, $(\nu, I \cup \{b\})$ is obtained from (μ, I) by a move of Type (1) and $f_\mu \geq f_\nu \geq f_{\mu'}$. It follows from our hypothesis that either $\nu = \mu$ or $\nu = \mu'$. Thus, either this move replaces μ by μ' or it leaves μ unchanged and decreases the cardinality of $I' - I$. We can continue applying moves of Type (1) in this manner until $I' \subseteq I$.

Now suppose that $I' \subseteq I$. If for some $b \in I'$ we have $f_\mu(b) > f_{\mu'}(b)$, define the function f by setting $f(a) = f_\mu(a)$ for all $a \in I - \{b\}$, $f(b) = f_{\mu'}(b)$, and requiring that f be

harmonic elsewhere. Again it is clear by Proposition 3.4.1 that $f = f_\nu$ for a point $\nu \in \mathfrak{h}_\mathbb{R}$ of Atiyah-Bott type for c. Since f_μ and f_ν are harmonic except at I, by Lemma 3.2.5 $f_\mu > f_\nu$. Since $I' \subseteq I$, both f_ν and $f_{\mu'}$ are harmonic except at I. Since $f_\mu \geq f_{\mu'}$, it follows that $f_\nu|I \geq f_{\mu'}|I$. Applying Lemma 3.2.5 once again, we see that $f_\nu \geq f_{\mu'}$. We can then obtain (ν, I) from (μ, I) by by a move of Type (3). Since $f_\mu > f_\nu \geq f_{\mu'}$, it follows from our hypothesis that $\nu = \mu'$. Thus, (μ', I) is obtained from (μ, I) by a single move of Type (3). Thus, this allows us to arrange that $I' \subseteq I$ and that $f_\mu|I' = f_{\mu'}|I'$. Then, we can obtain μ' from μ by a single move of Type (2). □

4.2 Bundle deformations

In this subsection, we shall concentrate on the case $g(C) = 1$. Here is the result which allows us to cover the moves of Theorem 4.1.1 by bundle moves.

Theorem 4.2.1. *Suppose that $g(C) = 1$. Fix $c \in \pi_1(G)$. Suppose (μ, I) and (μ', I') are of Atiyah-Bott type for some c, $f_\mu \geq f_{\mu'}$, and that (μ', I') is obtained from (μ, I) by one of the three moves described in Theorem 4.1.1. Let Ξ be a holomorphic P^I-bundle over C such that $\eta = \Xi/U^I$ is a semistable L^I-bundle whose Atiyah-Bott point is μ. Then there is an arbitrarily small deformation of $\Xi \times_{P^I} G$ to a G-bundle Ξ' which has a reduction $\Xi_{P^{I'}}$ over $P^{I'}$ such that $\eta' = \Xi_{P^{I'}}/U^{I'}$ is a semistable L'-bundle with Atiyah-Bott point μ'.*

Proof. We shall consider each type of move separately. We begin with two general lemmas.

Lemma 4.2.2. *Let H_1 and H_2 be connected linear algebraic groups over $\mathbb{C}$. Suppose that $p\colon H_1 \to H_2$ is a surjective homomorphism. Let Ξ_1 be a holomorphic H_1-bundle over a curve C of arbitrary genus and let Ξ_2 be the holomorphic H_2 bundle $\Xi_1 \times_{H_1} H_2$. Then any small deformation of Ξ_2 can be covered by a small deformation of Ξ_1.*

Proof. The tangent space to the deformations of Ξ_1 is given by $H^1(C; \mathrm{ad}_{H_1} \Xi_1)$. The tangent space to the space of deformations of Ξ_2 is given by $H^1(C; \mathrm{ad}_{H_2} \Xi_2)$. The natural map of vector bundles $\mathrm{ad}_{H_1} \Xi \to \mathrm{ad}_{H_2} \Xi$ is surjective, and hence the map $H^1(C; \mathrm{ad}_{H_1} \Xi) \to H^1(C; \mathrm{ad}_{H_2} \Xi_2)$ is surjective. From this the result follows. □

The following lemma will be used for holomorphic bundles, but of course a similar result holds in the C^∞ category.

Lemma 4.2.3. *Suppose that G_1, G_2, H are complex Lie groups and that $\phi_i\colon G_i \to H$ are homomorphisms with ϕ_1 surjective. Let ζ be a holomorphic principal G_1-bundle over a complex manifold X, and suppose that $\zeta \times_{G_1} H$ is isomorphic to a bundle of the form $\zeta' \times_{G_2} H$, where ζ' is a holomorphic G_2-bundle. Then there exists a holomorphic $G_1 \times_H G_2$-bundle ζ'', such that ζ is isomorphic to $\zeta'' \times_{(G_1 \times_H G_2)} G_1$.*

Proof. Choose an open cover $\{\mathcal{U}_i\}$ of X and trivializations of ζ and ζ' over each $\mathcal{U}_i$. Suppose that g_{ij} are the transition functions for ζ and g'_{ij} are those for ζ'. There exists a 0-cochain $\{h_i\}$ with values in H such that $h_i\phi_1(g_{ij})h_j^{-1} = \phi_2(g'_{ij})$. After shrinking the open cover, we can assume since ϕ_1 is surjective that the functions h_i lift to functions $\widetilde{h}_i \in G_1$ with $\phi_1(\widetilde{h}_i) = h_i$. Using the 0-cochain $\{\widetilde{h}_i\}$ to modify $\{g_{ij}\}$, we may then assume that the g_{ij} satisfy: $\phi_1(g_{ij}) = \phi_2(g'_{ij})$. Thus, $(g_{ij}, g'_{ij}) \in G_1 \times_H G_2$. Clearly, $\{(g_{ij}, g'_{ij})\}$ is a 1-cocycle, and the bundle ζ'' which it defines satisfies: $\zeta'' \times_{(G_1\times_H G_2)} G_1 \cong \zeta$. $\square$

Returning to the proof of Theorem 4.2.1, we begin with moves of Type (2), the simplest to describe. This case relies on the following elementary lemma which is implicit in [5] and [1]:

Lemma 4.2.4. *Let L be a reductive group. Every holomorphic L-bundle over C has an arbitrarily small deformation to a semistable L-bundle.*

Proof. Fix a C^∞ L-bundle η over C. The $(0,1)$-connections on η which determine a semistable holomorphic L bundle structure are a non-empty open subset of the space of all $(0,1)$-connections on η and hence form a dense open subset. $\square$

For a move of Type (2) $I' \subseteq I$ and $f_{\mu'}|I' = f_\mu|I'$. Thus, $L^I \subseteq L^{I'}$ and $P^I \subseteq P^{I'}$. By Lemma 4.2.4 the $L^{I'}$-bundle $\eta \times_{L^I} L^{I'}$ has an arbitrarily small deformation to a semistable $L^{I'}$-bundle η'. The Atiyah-Bott point of η' is μ'. Now apply Lemma 4.2.2 to the surjection $P^{I'} \to L^{I'}$ and the bundle $\Xi \times_{P^I} P^{I'}$ to produce an arbitrarily small deformation of the $P^{I'}$-bundle $\Xi \times_{P^I} P^{I'}$ to a $P^{I'}$-bundle Ξ' with $\Xi'/U^{I'}$ isomorphic to η'. Viewing this deformation of $P^{I'}$-bundles as giving a deformation of G-bundles exhibits the required deformation for a move of Type (2).

Now we turn to a move of Type (1). This time $I' = I \cup \{a\}$, so that $L^{I'} \subseteq L^I$ and $P^{I'} \subseteq P^I$. Let $P \subseteq L^I$ be the maximal parabolic $P^{I'} \cap L^I$. Its Levi factor is $L^{I'}$. Denote by U its unipotent radical. First let us consider the case where $f_{\mu'}$ is strictly subharmonic at a. This means that if η' is a semistable $L^{I'}$-bundle with Atiyah-Bott point μ', then the Harder-Narasimhan parabolic for the L^I-bundle $\eta' \times_{L^{I'}} L^I$ is P_-, the opposite parabolic to P. The relevant lemma for this case is the following.

Lemma 4.2.5. *Suppose that $g(C) = 1$. Let M be a reductive group and let Υ be a semistable M-bundle over C. Let $Q \subseteq M$ be a parabolic subgroup with Levi factor M_1 and unipotent radical V. Let Q_- be the opposite parabolic in M, and let V_- be its unipotent radical. Suppose that τ is a semistable M_1-bundle over C such that the Harder-Narasimhan parabolic of $\tau \times_{M_1} M$ is Q_-, and such that $\tau \times_{M_1} M$ and Υ are C^∞ isomorphic. Then there is an arbitrarily small deformation of Υ to a bundle of the form $\Upsilon_Q \times_Q M$ where Υ_Q is a holomorphic Q-bundle such that Υ_Q/V is semistable and is C^∞-isomorphic to τ.*

Proof. Let $\mathfrak{v}$, $\mathfrak{q}$, $\mathfrak{m}$, $\mathfrak{v}_-$ be the Lie algebras of V, Q, M, and V_- respectively. The direct sum decomposition $\mathfrak{m} = \mathfrak{q} \oplus \mathfrak{v}_-$ is preserved by the action of M_1. Thus

$$\mathrm{ad}_M(\tau \times_{M_1} M) \cong \mathrm{ad}_Q(\tau \times_{M_1} Q) \oplus \mathfrak{v}_-(\tau).$$

Since Q_- is the Harder-Narasimhan parabolic for $\tau \times_{M_1} M$, the bundle $\mathfrak{v}_-(\tau)$ is a direct sum of semistable bundles of positive degrees. Since $g(C) = 1$, it follows from stability and Serre duality that $H^1(C; \mathfrak{v}_-(\tau)) = 0$. Thus the natural map

$$H^1(C; \mathrm{ad}_Q(\tau \times_{M_1} Q)) \to H^1(C; \mathrm{ad}_M(\tau \times_{M_1} M))$$

is surjective. Since C is a curve, all deformations are unobstructed, and the map from the deformation space of the Q-bundle $\tau \times_{M_1} Q$ to that of the M-bundle $\tau \times_{M_1} M$ is a submersion. Thus every arbitrarily small deformation of the M-bundle $\tau \times_{M_1} M$ arises from an arbitrarily small deformation of the Q-bundle $\tau \times_{M_1} Q$. In particular, there is an arbitrarily small deformation of the Q-bundle $\tau \times_{M_1} Q$ whose associated M-bundle is semistable.

The set of all semistable L-bundles C^∞ isomorphic to τ may be parametrized by an irreducible scheme, in the sense that there exists an irreducible scheme S_0 and an L-bundle over $S_0 \times C$ whose restriction to every slice $\{s\} \times C$ is semistable, and moreover such that every semistable L-bundle C^∞ isomorphic to τ arises in this way. By the results of [4, Appendix], since by Riemann-Roch $\dim H^1(C; \mathfrak{v}(\tau))$ is independent of τ, there is an irreducible scheme S_1, fibered in affine spaces over S_0, which parametrizes all Q-bundle deformations of $\tau \times_{M_1} Q$. By the above, there is a nonempty open subset S_1^{ss} of S_1 and a dominant morphism from S_1^{ss} to the moduli space of all semistable M-bundles of the same topological type as Υ, and this moduli space is irreducible. From this, the result follows. $\square$

Applying this lemma with $M = L^I$, $Q = P$, $\Upsilon = \eta$, we see that there is an arbitrarily small deformation of the L^I-bundle η to a bundle of the form $\eta_P \times_P L^I$ where η_P is a P-bundle and η_P/U is a semistable $L^{I'}$-bundle with Atiyah-Bott point μ'.

Using Lemma 4.2.2 for the surjection $P^I \to L^I$, we produce an arbitrarily small deformation of Ξ to a bundle Ξ' whose reduction modulo U^I is isomorphic to $\eta_P \times_P L^I$. Since $P^{I'} \subseteq P^I$ is the preimage of P under the natural projection $P^I \to L^I$, and hence $P^{I'} = P^I \times_{L^I} P$, it follows from Lemma 4.2.3 that Ξ' can be written as $\Xi_{P^{I'}} \times_{P^{I'}} P^I$ for some $P^{I'}$-bundle $\Xi_{P^{I'}}$ whose reduction modulo $U^{I'}$ is η_P. This completes the discussion of the move of Type (1) in the case when $f_{\mu'}$ is strictly subharmonic at a.

Now assume that $f_{\mu'}$ is harmonic at a. This means that $\mu = \mu'$.

Lemma 4.2.6. *Suppose that $g(C) = 1$. Let M be a reductive group and let Υ be a semistable M-bundle over C. Let $Q \subseteq M$ be a parabolic subgroup with Levi factor M_1 and unipotent radical V. Then there is an arbitrarily small deformation of Υ to a bundle of the form $\Upsilon' \times_{M_1} M$ where Υ' is a semistable M_1-bundle whose Atiyah-Bott point is equal to that of Υ under the inclusion of the center of M into the center of M_1.*

Proof. Let $\mathfrak{v}$ be the Lie algebra of V and $\mathfrak{v}_-$ the Lie algebra of the opposite unipotent radical. Let $Z(M_1)$ denote the identity component of the center of M_1. The M_1-module $\mathfrak{v}$ decomposes as a direct sum $\bigoplus_\chi \mathfrak{v}_\chi$ where $\chi \in \widehat{Z(M_1)}$ are characters vanishing on the identity component of the center of M. No χ is trivial, since $\mathfrak{v}$ is a direct sum of root

spaces $\mathfrak{m}^\alpha$ corresponding to roots which are not trivial on $Z(M_1)$. Since $Z(M_1)$ is the center of M_1, there is an action of the group of holomorphic $Z(M_1)$-bundles on the space of holomorphic M_1-bundles. We denote this action by $(\lambda,\Gamma)\mapsto \Gamma\otimes\lambda$. If $c_1(\lambda)=0$ and if Γ is semistable, then $\Gamma\otimes\lambda$ is a semistable holomorphic M_1-bundle which is C^∞-isomorphic to Γ. Fix Γ a semistable M_1-bundle whose Atiyah-Bott point is that of Υ under the inclusion $Z(M)\subseteq Z(M_1)$. For any $\chi\in\widehat{Z(M_1)}$ trivial on $Z(M)$, consider the vector bundle $\mathfrak{v}_\chi(\Gamma)$. This is a semistable vector bundle of degree zero over the genus one curve C. Of course, $\mathfrak{v}_\chi(\Gamma\otimes\lambda)=\mathfrak{v}_\chi(\Gamma)\otimes\chi(\lambda)$. For generic choices of λ with $c_1(\lambda)=0$, the cohomology group $H^1(C;\mathfrak{v}_\chi(\Gamma\otimes\lambda))$ vanishes for all characters χ, and hence for such generic λ we have $H^1(C;\mathfrak{v}(\Gamma\otimes\lambda))=0$. By duality, $H^1(C;\mathfrak{v}_-(\Gamma\otimes\lambda))=0$ for generic such λ. For such λ,

$$H^1(C;\mathrm{ad}_{M_1}(\Gamma\otimes\lambda))=H^1(C;\mathrm{ad}_M((\Gamma\otimes\lambda)\times_{M_1}M)).$$

Consider the set of all pairs (Γ,λ) for which $H^1(C;\mathfrak{v}(\Gamma\otimes\lambda))=0$. As in the proof of the previous lemma, there is an irreducible scheme S parametrizing (possibly many-to-one) the isomorphism classes of such pairs and a dominant map from S to the moduli space of semistable M-bundles C^∞-isomorphic to Υ. Hence, the generic such M-bundle can be written as $\Upsilon'\times_{M_1}M$ with Υ' a semistable M_1-bundle with the same Atiyah-Bott point as Υ. □

Applying this lemma with $M=L^I$, $Q=P$, $\Upsilon=\eta$, we see that there is an arbitrarily small deformation of the L^I-bundle η to a bundle of the form $\eta_P\times_P L^I$ where η_P/U is a semistable $L^{I'}$-bundle with Atiyah-Bott point μ. Applying Lemma 4.2.2 and Lemma 4.2.3 as before completes the discussion in this case.

Notice that we did not use the hypothesis that there were no Atiyah-Bott points of type c strictly between μ and μ'. We include this hypothesis in order to handle the case of genus greater than one below.

Now we consider a move of Type (3). The relevant result is the following.

Theorem 4.2.7. *Let M be a reductive group and let α be a simple root of M. Let $Q=Q^\alpha$ be the corresponding maximal parabolic, V its unipotent radical and M_1 its Levi factor. Suppose that η is a semistable M_1-bundle with Atiyah-Bott point μ with $f_\mu(\alpha^\vee)=q$. Then there is an arbitrarily small M-deformation of $\eta\times_{M_1}M$ to a M-bundle of the form $\eta_Q\times_Q M$ where η_Q is a Q-bundle with η_Q/V a semistable M_1-bundle whose Atiyah-Bott point μ' satisfies $f_{\mu'}(\alpha^\vee)=q-1$.*

The proof of this theorem involves some new ideas and we postpone it to the next section. We show how this result implies Theorem 4.2.1 for moves of Type (3). Set $M=L^{I-\{a\}}$ and $P=P^{I-\{a\}}$ and let U be the unipotent radical of P. We let $Q=P^I\cap M$ and $M_1=L^I$. Since $f_\mu(a)-f_{\mu'}(a)$ is a positive integer, applying Theorem 4.2.7 repeatedly produces an arbitrarily small deformation of the M-bundle $\eta\times_{L^I}M$ to a bundle of the form $\eta_Q\times_Q M$ where η_Q is a semistable Q-bundle such that the Atiyah-Bott point of $\eta'=\eta_Q/V$ is μ'. Applying Lemma 4.2.2 produces an arbitrarily small deformation of the P-bundle $\Xi\times_{P^I}P$ to a P-bundle of the form Ξ_P where Ξ_P/U is isomorphic to $\eta_Q\times_Q M$. It follows as before

from Lemma 4.2.3 that Ξ_P reduces to a P^I-bundle Ξ_{P^I} with Ξ_{P^I}/U^I isomorphic to η'. This concludes the proof of Theorem 4.2.1. $\square$

Proof of Theorem 2.4.6. Theorem 2.4.6 is an immediate consequence of Theorem 4.2.1 and Theorem 4.1.1. $\square$

4.3 The case of higher genus

Here is the (weaker) statement in the case of higher genus.

Theorem 4.3.1. *Suppose that $g(C) \geq 2$. Let (μ, I) and (μ', I') be of Atiyah-Bott type for c, $f_\mu \geq f_{\mu'}$, and suppose that (μ', I') is obtained from (μ, I) by one of the three elementary moves described in Theorem 4.1.1. Then there exists a holomorphic P^I-bundle Ξ such that $\eta = \Xi/U^I$ is a semistable L^I-bundle whose Atiyah-Bott point is μ and an arbitrarily small deformation of $\Xi \times_{P^I} G$ to a G-bundle Ξ' which has a reduction $\Xi_{P^{I'}}$ over $P^{I'}$ with $\eta' = \Xi_{P^{I'}}/U^{I'}$ a semistable L'-bundle with Atiyah-Bott point μ'.*

Proof. The proof given above for moves of Types (2) and (3) applies equally well for curves of higher genus. For the case of moves of Type (1) we further divided into the case when $f_{\mu'}$ was strictly subharmonic at a and the case when $f_{\mu'}$ was harmonic at a. The proof in this case when $f_{\mu'}$ is harmonic reduces to the following elementary fact. Given two reductive groups $L' \subseteq L$ and a semistable L'-bundle η' whose Atiyah-Bott point lies in the center of L, the L-bundle $\eta' \times_{L'} L$ is semistable.

The remaining case is where $f_{\mu'}$ is strictly subharmonic at a. Let $L^I = L$. Let η_0 be a C^∞ L-bundle with Atiyah-Bott point μ and such that $\eta_0 \times_L G \cong \xi_0$. The set $\Delta - I$ is a set of simple roots for L and determines a fundamental Weyl chamber $\overline{C}_0(L)$ in $\mathfrak{h}$ for the Weyl group of L. This chamber contains $\overline{C}_0$ but will be strictly larger than it if $I \neq \emptyset$. Since f_μ and $f_{\mu'}$ are subharmonic at a and harmonic on $\Delta^\vee - (I \cup \{a\})$, it follows that μ, μ' lie in $-\overline{C}_0(L)$. Our hypothesis is that there is no point ν of Atiyah-Bott type for c (and the group G) with $f_\mu > f_\nu > f_{\mu'}$. Hence there is no point ν of Atiyah-Bott type for $c_1(\eta_0)$ (and the group L) with $f_\mu > f_\nu > f_{\mu'}$. Of course, under the Atiyah-Bott ordering for L we have $\mu' > \mu$ since μ indexes the stratum of semistable L-bundles. According to Lemma 3.2.6, applied to the group L, this implies that there is no point of Atiyah-Bott type for $c_1(\eta_0)$ with $\mu' > \nu > \mu$ in the Atiyah-Bott ordering.

Let $P \subseteq L$ be the maximal parabolic subgroup whose Levi factor is L', and let $U \subseteq P$ be its unipotent radical and $\mathfrak{u}$ its Lie algebra. We denote by P_- the opposite parabolic, by U_- its unipotent radical and by $\mathfrak{u}_-$ the Lie algebra of this opposite unipotent radical. Begin with a semistable L'-bundle η' with Atiyah-Bott point μ' and such that $\eta \times_{L'} L$ is C^∞ isomorphic to η_0. The Harder-Narasimhan parabolic for η' is P_-. Thus the tangent space to the stratum containing η' is $H^1(C; \mathrm{ad}_{P_-}(\eta'))$ and the normal space is $H^1(C; \mathfrak{u}(\eta'))$. The bundle $\mathfrak{u}(\eta')$ is a direct sum of semistable vector bundles of negative degrees, and so $H^0(C; \mathfrak{u}(\eta')) = 0$. Thus, by Riemann-Roch $H^1(C; \mathfrak{u}(\eta')) \neq 0$. This means that there is

an arbitrarily small P-deformation Ξ_P of η' such that $\Xi_P \times_P L$ is not contained in the stratum containing $\eta' \times_{L'} L$. Let $\Xi = \Xi_P \times_P L$. According to the theorem of Atiyah-Bott, this means that the Atiyah-Bott point $\mu' > \mu(\Xi)$ in the Atiyah-Bott ordering. Of course, since μ is the Atiyah-Bott point of semistable L-bundles we have $\mu(\Xi) \geq \mu$. It now follows from the discussion in the previous paragraph that $\mu(\Xi) = \mu$, i.e., that Ξ is a semistable L-bundle of the topological type of η.

Thus, we have produced an L-bundle Ξ which is semistable and whose Atiyah-Bott point is μ which reduces to a P-bundle Ξ_P whose associated $L^{I'}$-bundle Ξ_P/U is isomorphic to η' and hence is semistable with Atiyah-Bott point μ'. This completes the proof of the theorem in this last case. $\square$

Proof of Theorem 2.4.7. Theorem 2.4.7 is an immediate consequence of Theorem 4.3.1 and Theorem 4.1.1 as well as Theorem 2.4.4. $\square$

4.4 An example

Let us give an example to show that the stronger theorem which we proved in the case of genus one does not hold for any curve of higher genus. The example will be for the group $SL(3)$, but surely similar examples can be constructed for any reductive group whose derived subgroup is of rank at least two. Fix a smooth curve C of genus at least 2. Consider a rank three vector bundle with trivial determinant which is given as an extension

$$0 \to \lambda \to V \to W \to 0,$$

where λ is a line bundle of degree 2 and W is a **stable** rank-two bundle of degree -2. Note that such bundles exist on any curve of genus greater than one. Viewing the Cartan subalgebra of $SL(3)$ as the subspace of $\mathbb{C}^3$ of triples (a,b,c) with $a+b+c=0$, with the Weyl chamber defined by $a \geq b \geq c$, the Atiyah-Bott point of V is $\mu = (2,-1,-1)$. Consider the Atiyah-Bott point $\mu' = (1,0,-1)$. Clearly, $\mu > \mu'$. The stratum $\mathcal{C}_{\mu'}$ consists of all rank three vector bundles with trivial determinant and a three-step filtration whose successive quotients are line bundles of degrees $1, 0, -1$, respectively. If there were an arbitrarily small deformation of V to a bundle with such a filtration, then by upper semi-continuity and the compactness of the space of line bundles of degree -1 on C, then there would exist a non-zero map from V to a line bundle of degree -1. Since $\deg \lambda = 2$, the restriction of this map to λ is trivial. Thus, there would be an induced non-zero map from W to a line bundle of degree -1, contradicting the stability of W. This shows that the bundle V which is in the stratum $\mathcal{C}_\mu$ is not in the closure of the stratum $\mathcal{C}_{\mu'}$. Of course, we can deform V within its stratum to a bundle V' which sits in an exact sequence

$$0 \to \lambda \to V' \to W' \to 0$$

with W' properly semistable. Such a bundle V' has an arbitrarily small deformation to a bundle contained in $\mathcal{C}_{\mu'}$, so that indeed the closure of $\mathcal{C}_{\mu'}$ meets $\mathcal{C}_\mu$.

5 Elementary modifications

5.1 Statement of the main theorem

The main goal in this section is to prove Theorem 4.2.7. For simplicity, we change notation in that theorem, so that M becomes G, Q becomes P, and so on. In fact, we show the following:

Theorem 5.1.1. *Let $P = P^\alpha$ be a standard maximal parabolic subgroup of G, with Levi subgroup L. Let η be a semistable L-bundle and let $\mu = \mu(\eta)$ be such that $f_\mu(\alpha^\vee) = q$. Let P_- denote the opposite parabolic to P and let U_- be the unipotent radical of P_-. Then, possibly after replacing η by an arbitrarily small deformation, there exists a P_--bundle ξ such that $\xi/U_- = \eta$ and such that $\xi \times_{P_-} G \cong \xi_P \times_P G$, where ξ_P is a P-bundle such that ξ_P/U is a semistable L-bundle whose Atiyah-Bott point μ' satisfies $f_{\mu'}(\alpha^\vee) = q - 1$.*

Proof of Theorem 4.2.7. It is a standard argument (see for example [4, Section 4]) that there exists a holomorphic G-bundle Υ over $\mathbb{C} \times C$ such that $\Upsilon|\{0\} \times C \cong \eta \times_L G$ and such that, for all $t \neq 0$, $\Upsilon|\{0\} \times C \cong \xi \times_{P_-} G \cong \xi_P \times_P G$. Thus the G-bundle $\xi_P \times_P G$ is an arbitrarily small deformation of $\eta \times_L G$, as claimed. ☐

5.2 Definition of elementary modifications

Let L be a reductive group and let η be a principal L-bundle. Let $p_0 \in C$, and let $\mathcal{U} = \{\mathcal{U}_i : i = 1, \dots, n\}$ be an open cover of C such that $p_0 \in \mathcal{U}_1$ and $p_0 \notin \mathcal{U}_i$ for $i > 1$. Fix a trivialization of η with respect to the open cover $\mathcal{U}$ and let g_{ij}, $i < j$, be the transition functions. By convention, this means that there exist local trivializations s_i of $\eta|\mathcal{U}_i$ and s_j of $\eta|\mathcal{U}_j$, and $s_i \circ s_j^{-1}|\mathcal{U}_i \cap \mathcal{U}_j$ is left multiplication by g_{ij}. If we are given local sections σ_i with $s_i(\sigma_i(x)h) = (x, h)$, then $\sigma_i = g_{ij}^{-1}\sigma_j$. Choose a small disk $\mathcal{U}_0 \subseteq \mathcal{U}_1$ containing p_0, and such that $\mathcal{U}_0 \cap \mathcal{U}_i = \emptyset$ for $i > 1$. Now define a new open cover $\mathcal{U}' = \{\mathcal{U}_0, \mathcal{U}_1 - \{p_0\}, \mathcal{U}_2, \dots, \mathcal{U}_n\}$. The transition functions for η relative to the open cover $\mathcal{U}'$ are equal to g_{ij} for $i, j \geq 1$, and $g_{01} = 1$. Note that $\mathcal{U}_0 \cap (\mathcal{U}_1 - \{p_0\}) = \mathcal{U}_0 - \{p_0\}$ is a punctured disk.

Definition 5.2.1. Suppose that $\varphi\colon \mathbb{C}^* \to L$ is a 1-parameter subgroup, in other words a homomorphism from $\mathbb{C}^*$ to L. With $\mathcal{U}_0$ as above, fix an isomorphism of $\mathcal{U}_0$ to a neighborhood of $0 \in \mathbb{C}$ sending p_0 to 0, and use this isomorphism to view φ as a function from $\mathcal{U}_0 - \{p_0\}$ to L. The coordinate t on $\mathbb{C}$ thus defines a coordinate, also denoted t, on $\mathcal{U}_0$. Define the *elementary modification* η_φ of η at p_0 (with respect to φ and the given trivialization of η) to be the L-bundle given by the following transition functions g'_{ij}, $i < j$, with respect to the open cover $\mathcal{U}'$: For $(i, j) \neq (0, 1)$, $g'_{ij} = g_{ij}$, and $g'_{01} = \varphi$.

The motivation for this definition is as follows. Two bundles which differ by an elementary modification should be isomorphic away from p_0 and hence birationally isomorphic. By Iwahori's theorem, every double coset in the space $G(\mathbb{C}[[t]])\backslash G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ is represented by a 1-parameter subgroup, and so it is natural to use these as the gluing maps for the new bundle.

We shall not try to describe the way η_φ changes for different choices of the trivialization or the isomorphism from $\mathcal{U}_0$ to a neighborhood of $0 \in \mathbb{C}$. It is easy to see that different choices may lead to non-isomorphic bundles. On the other hand, the topological type of η_φ is determined by that of η and by φ. More precisely, we have:

Lemma 5.2.2. *With notation as above, $\mu(\eta_\varphi) = \mu(\eta) + \pi\varphi_*(1)$, where π is the projection from $\mathfrak{h}$ to $\mathfrak{z}_G$.*

Proof. It suffices to show that, for all characters χ of L, we have $\deg \chi(\eta_\varphi) = \deg \chi(\eta) + n$, where n is the integer such that $\chi \circ \varphi(t) = t^n$. But clearly $\chi(\eta_\varphi) = \chi(\eta)_{\chi\circ\varphi}$, and we are reduced to the case of a line bundle, i.e. $L = \mathbb{C}^*$ and $\varphi(t) = t^n$. In this case, it is easy to see that, if λ is the line bundle over C corresponding to η, then the line bundle corresponding to η_φ is $\lambda \otimes \mathcal{O}_C(np_0)$, and then the statement about degrees is clear. $\square$

In practice, the groups L will arise as Levi subgroups of G, and so a 1-parameter subgroup of L is also a 1-parameter subgroup of G. Thus an elementary modification of an L-bundle η is also an elementary modification of the G-bundle $\eta \times_L G$. It will be important to know when this construction does not change the topological type of $\eta \times_L G$.

Lemma 5.2.3. *Let ξ be a holomorphic G-bundle, and let φ be a 1-parameter subgroup of G which lifts to the universal cover $\widetilde{G}$. Fix an open cover and transition functions for ξ, and define ξ_φ as above. Then $c_1(\xi_\varphi) = c_1(\xi)$, in other words the bundles ξ_φ and ξ are isomorphic as C^∞ bundles.*

Proof. We may assume that the cover $\mathcal{U}$ was chosen so that each g_{ij}, $(i,j) \neq (0,1)$, lifts to $\widetilde{g}_{ij} \in \widetilde{G}$. By hypothesis, φ lifts to $\widetilde{\varphi} \in \widetilde{G}$. Thus, we have lifted the transition functions g'_{ij} of ξ_φ to a collection $\widetilde{g}'_{ij} \in \widetilde{G}$. By definition, $c_1(\xi_\varphi)$ is the coboundary of $\{\widetilde{g}'_{ij}\}$, viewed as an element of $H^2(C; Z(G))$, and similarly for $c_1(\xi)$. But clearly, since there are no triple intersections of the $\mathcal{U}_i$ involving $\mathcal{U}_0$, we have $c_1(\xi_\varphi) = c_1(\xi)$. $\square$

In fact, the condition on φ in the lemma is both necessary and sufficient, although we shall not use this fact.

Returning to the case of an L-bundle, we shall show that we may assume that the bundle η_φ is in general semistable.

Lemma 5.2.4. *With η_φ as above, there exists an arbitrarily small deformation η_s of η and of the trivializations so that $(\eta_s)_\varphi$ is semistable.*

Proof. By Lemma 4.2.4, there is an arbitrarily small deformation of $(\eta_\varphi)_s$ which is semistable. If g'_{ij} are the transition functions for η_φ as defined above, this means that we can find transition functions $g'_{ij}(s)$, depending on s in a small disk about 0 in $\mathbb{C}$, such that $g'_{ij}(0) = g'_{ij}$, $0 \leq i < j$, and such that the bundle whose transition functions are $g'_{ij}(s)$ is semistable for all $s \neq 0$. Define $g_{ij}(s) = g'_{ij}(s)$ for $(i,j) \neq (0,1)$, and $g_{01}(s) =$

$g'_{01}(s)\cdot\varphi^{-1}$. Clearly the functions $g_{ij}(s)$ satisfy the cocycle condition. For $(i,j)\neq(0,1)$, $g_{ij}(0)=g'_{ij}(0)=g_{ij}$, and $g_{01}(0)=g'_{01}(0)\cdot\varphi^{-1}=g'_{01}\cdot\varphi^{-1}=1=g_{01}$. Thus the functions $g_{ij}(s)$ define a deformation η_s of η, possibly trivial, and $(\eta_s)_\varphi=(\eta_\varphi)_s$ is semistable for all $s\neq 0$. ∎

Corollary 5.2.5. *Let G be a reductive group and let $P=P^\alpha$ be a standard maximal parabolic subgroup of G, with Levi subgroup L. Let η be a semistable L-bundle and let $\mu=\mu(\eta)$ be such that $f_\mu(\alpha^\vee)=q$. Let φ be the 1-parameter subgroup of G such that $\varphi_*(1)=-\alpha^\vee$. Possibly after replacing η by an arbitrarily small deformation, we may assume that:*

(i) *η_φ is semistable;*

(ii) *$\eta_\varphi\times_L G$ is C^∞ isomorphic to $\eta\times_L G$;*

(iii) *The Atiyah-Bott point of η_φ is the unique $\mu'\in\mathfrak{z}_L$ such that $f_{\mu'}(\alpha^\vee)=f_\mu(\alpha^\vee)-1$.*

Proof. That we can arrange (i) after an arbitrarily small deformation of η follows from Lemma 5.2.4. Since φ lifts to $\widetilde{G}$, (ii) follows from Lemma 5.2.3. Part (iii) follows from Lemma 5.2.2. ∎

5.3 Writing the elementary modification as an extension

Our goal now is to prove:

Proposition 5.3.1. *Let η, the open cover $\mathcal{U}'$, and the transition functions g_{ij} be as in the beginning of this section. Fix a nonzero $X\in\mathfrak{g}^{-\alpha}$. Let $\{u_{ij}\}$ be the 1-cochain with respect to the open cover $\mathcal{U}'$ with values in the sheaf $U_-(\eta)$ defined as follows: $u_{ij}=1$ for $(i,j)\neq(0,1)$, and $u_{01}=\exp(t^{-1}X)$, where t is the coordinate on $\mathcal{U}_0$ defined by the inclusion of $\mathcal{U}_0$ in $\mathbb{C}$. Then $\{u_{ij}\}$ is a cocycle. Let ξ be the P_--bundle defined by the 1-cocycle $\{g_{ij}u_{ij}\}$. Then the bundle $\xi\times_{P_-}G$ has a reduction to P, such that the associated L-bundle is isomorphic to η_φ.*

Proof. Since there are no nonempty triple intersections involving $\mathcal{U}_0$ and $\mathcal{U}_1$, $\{u_{ij}\}$ is vacuously a 1-cocycle. To prove the rest of the result, we shall show: for all $i<j$, there exists $v^+_{ij}\colon\mathcal{U}_i\cap\mathcal{U}_j\to U$ and, for all i, there exists $u_i\colon\mathcal{U}_i\to G$ such that

$$u_ig'_{ij}v^+_{ij}u_j^{-1}=g_{ij}u_{ij}.$$

For this says that the 1-cocycles $g'_{ij}v^+_{ij}$ and $g_{ij}u_{ij}$ are cohomologous. Clearly, the cocycle $g'_{ij}v^+_{ij}$ defines a P-bundle ξ_P whose associated L-bundle is η_φ as required.

We can rewrite the above condition as $g_{ij}u_i^{g_{ij}}v^+_{ij}u_j^{-1}=g_{ij}$ if $i<j$ and $i\neq 0$, or in other words $u_i^{g_{ij}}v^+_{ij}u_j^{-1}=1$. This condition is automatically satisfied if $u_i\in U$ for $i\geq 1$ by

setting $v_{ij}^+ = (u_i^{g_{ij}})^{-1}u_j$. Thus for example we can take $u_i = 1$ for $i \geq 1$ and set $v_{ij}^+ = 1$ for $1 \leq i \leq j$. For $(i,j) = (0,1)$ we get the single condition $u_0\varphi v_{01}^+ u_1^{-1} = \exp(t^{-1}X)$ for some $v_{01}^+ \colon \mathcal{U}_0 \cap \mathcal{U}_1 \to U$. Assuming that we have chosen $u_1 = 1$, we seek a function $u_0 \colon \mathcal{U}_0 \to G$ such that

$$\varphi^{-1}u_0^{-1}\exp(t^{-1}X) = v_{01}^+ \in U.$$

Clearly, it suffices to solve this equation in the $SL(2)$ which is a subgroup of the universal cover of G and whose Lie algebra is $\mathfrak{g}^{-\alpha} \oplus \mathbb{C} \cdot \alpha^\vee \oplus \mathfrak{g}^\alpha$. In this copy of $SL(2)$, $\varphi = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$ and we may assume that $\exp(t^{-1}X) = \begin{pmatrix} 1 & 0 \\ t^{-1} & 1 \end{pmatrix}$. Taking $u_0^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & t \end{pmatrix}$, we have the equation

$$\varphi^{-1}u_0^{-1}\exp(t^{-1}X) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

as required. □

Proof of Theorem 5.1.1. Theorem 5.1.1 follows immediately from Corollary 5.2.5 and Proposition 5.3.1.

Remark 5.3.2. A very similar proof shows that we can begin with a P-bundle Ξ such that $\Xi/U \cong \eta$ and deform the G-bundle $\Xi \times_P G$ to a bundle of the form $\Xi' \times_P G$, where $\Xi'/U \cong \eta_\varphi$.

6 The minimally unstable strata over maximal parabolics

6.1 Definition of the minimally unstable strata

Definition 6.1.1. Let μ be an Atiyah-Bott point of type c which does not lie in $\mathfrak{z}_G$. We say that μ is *minimally unstable* if μ is minimal among all Atiyah-Bott points of type c which do not lie in $\mathfrak{z}_G$. Equivalently, the stratum $\mathcal{C}_\mu$ is minimally unstable if, for all $\mu' \neq \mu$ of type c, the closure of $\mathcal{C}_{\mu'}$ meets $\mathcal{C}_\mu$ if and only if $\mu' \in \mathfrak{z}_G$.

Thus, $\mathcal{C}_\mu$ is minimally unstable if and only if it consists of unstable bundles, and, for all ξ lying in $\mathcal{C}_\mu$, every small deformation of ξ either lies in $\mathcal{C}_\mu$ or is semistable.

We turn now to a detailed and explicit discussion of the minimally unstable strata. Not suprisingly, as the next lemma shows, these are always associated to maximal parabolic subgroups.

Lemma 6.1.2. *Let ξ be a C^∞-bundle over a curve C. Suppose that $\mu \in \overline{C}_0$ is a minimally unstable point of Atiyah-Bott type for ξ. Let $P(\mu)$ be the parabolic subgroup determined by μ, i.e. $P(\mu) = P^{I(\mu)}$ where $I(\mu)$ consists of all $\alpha \in \Delta$ with the property that f_μ is not harmonic at $\alpha^\vee$. Then $P(\mu)$ is a maximal parabolic. Furthermore,* $0 < f_\mu(\alpha^\vee) \leq 1$.

Proof. By definition, $\#I \geq 1$, and so it suffices to show that $\#I = 1$. Choose $\alpha \in I$, and let $f\colon \Delta^\vee \to \mathbb{R}$ be define by $f(\alpha^\vee) = f_\mu(\alpha^\vee)$ and f is harmonic outside of α. By Lemma 3.2.4, $f \leq f_\mu$ and $f(\alpha^\vee) = f_\mu(\alpha^\vee) > 0$. Since f is harmonic except at α it follows from Proposition 3.2.3 that f is superharmonic. Clearly, by Lemma 3.4.1 $f = f_\nu$ for some point $\nu \in \overline{C}_0$ of Atiyah-Bott type for ξ and $f_\nu \leq f_\mu$, so that $\nu \leq \mu$ in the Atiyah-Bott ordering. By minimality, $\nu = \mu$. Thus $I(\mu) = I(\nu) = \{\alpha\}$. This proves the first statement in the lemma. To see the second, if $f_\mu(\alpha^\vee) > 1$, then there is a unique function $f\colon \Delta^\vee \to \mathbb{R}$ which is harmonic except at $\alpha^\vee$ and such that $f(\alpha^\vee = f_\mu(\alpha^\vee) - 1 > 0$. But then f is associated to a point μ' of Atiyah-Bott type for c such that $\mu' \in \overline{C}_0$, $\mu' < \mu$, and $\mu' \notin \mathfrak{z}_G$. This contradicts the choice of μ. □

Given c, there is exactly one unstable stratum of the type considered in the last lemma for each $\alpha \in \Delta$. We label the Atiyah-Bott point for the stratum associated to α by $\mu_{c,\alpha}$. In case $c = 1$, we set $\mu_{1,\alpha} = \mu_\alpha$. Our next task is to understand the partial ordering on the $\{\mu_{c,\alpha}\}_{\alpha\in\Delta}$.

Here is the result that allows us to determine the Atiyah-Bott partial order on these points.

Theorem 6.1.3. *Given $\alpha, \beta \in \Delta$, let f_α be a superharmonic function, harmonic except at $\alpha^\vee$, and similarly for f_β. Then $f_\alpha \leq f_\beta$ if and only if $f_\alpha(\alpha^\vee) \leq f_\beta(\alpha^\vee)$.*

Proof. Clearly, if $f_\alpha \leq f_\beta$, then $f_\alpha(\alpha^\vee) \leq f_\beta(\alpha^\vee)$.

To prove the converse, we use our previous results on harmonic and superharmonic functions. The Dynkin diagram of $\Delta^\vee - \{\alpha^\vee\}$ is a union of t connected components $C_1, \dots, C_t$. For $1 \leq i \leq t$, let $\Delta_i^\vee$ be the set of the vertices of C_i together with $\{\alpha^\vee\}$, and assume that $\beta^\vee \in \Delta_1^\vee$. For $i > 1$, both f_α and f_β are harmonic on $\Delta_i^\vee - \{\alpha^\vee\}$, and $f_\alpha(\alpha^\vee) \leq f_\beta(\alpha^\vee)$. It follows from Lemma 3.2.5 that $f_\alpha(\gamma^\vee) \leq f_\beta(\gamma^\vee)$ for all $\gamma^\vee \in \Delta_i^\vee, i \neq 1$.

Now consider the restrictions of f_α and f_β to $\Delta_1^\vee$. The function f_α is harmonic except at $\{\alpha^\vee\}$, and the function f_β is superharmonic on $\Delta_1^\vee$. Since $f_\alpha(\alpha^\vee) \leq f_\beta(\alpha^\vee)$, Lemma 3.2.4 implies that $f_\alpha(\gamma^\vee) \leq f_\beta(\gamma^\vee)$ for all $\gamma^\vee \in \Delta_1^\vee$. Hence, for all $\gamma^\vee \in \Delta^\vee$, $f_\alpha(\gamma^\vee) \leq f_\beta(\gamma^\vee)$. This concludes the proof of Theorem 6.1.3. □

6.2 The simply connected case: A characterization of the partial order

In this section, we assume that G is simple and simply connected. Then the strata of G-bundles whose Harder-Narasimhan parabolic is a maximal parabolic and which are of minimal positive degree with respect to the unique dominant character of this parabolic are indexed by the points μ_α, where μ_α is the unique point such that $\beta(\mu_\alpha) = 0$ for $\beta \neq \alpha$ (i.e., f_{μ_α} is harmonic except at $\alpha^\vee$) and $f_{\mu_\alpha}(\alpha^\vee) = 1$. By Theorem 6.1.3, $\mu_\alpha \leq \mu_\beta$ if and only if $f_{\mu_\beta}(\alpha^\vee) \geq 1$, in other words if and only if the coefficient $\varpi_\alpha(\mu_\beta)$ of $\alpha^\vee$ in μ_β, expressed as a sum of the simple coroots, is at least 1. The stratum $\mathcal{C}_{\mu_\alpha}$ consists of bundles $\xi \times_{P^\alpha} G$, where ξ is a P^α-bundle such that $\xi/U = \eta$ is a semistable L^α-bundle with $\deg \eta = 1$.

Definition 6.2.1. A simple root α is *special* if

(i) The Dynkin diagram associated to $\Delta - \{\alpha\}$ is a union of diagrams of type A;

(ii) The simple root α meets each component of the Dynkin diagram associated to $\Delta-\{\alpha\}$ at an end of the component;

(iii) The root α is a long root.

If R is of type A_n, then every simple root is special. All other irreducible root systems have a unique special simple root. It corresponds to the unique trivalent vertex if the Dynkin diagram is of type $D_n, n \geq 4$ or $E_n, n = 6, 7, 8$. For $R = C_n, n \geq 2$ or G_2, it is the long simple root. For $R = B_n, n \geq 2$ and F_4 it is the unique long simple root which is not orthogonal to a short simple root.

Let α be special. As in the proof of Theorem 6.1.3, suppose that the Dynkin diagram of $\Delta^\vee - \{\alpha^\vee\}$ is a union of t connected components $C_1, \dots, C_t$, and let $\Delta_i^\vee$ be the set of the vertices of C_i together with $\{\alpha^\vee\}$. Since the Dynkin diagram of $\Delta^\vee$ has at most one trivalent vertex, each $\Delta_i^\vee$ is a chain. We may uniquely label the coroots of $\Delta_i^\vee$ as $\beta_{i,1}^\vee, \dots, \beta_{i,n_i}^\vee = \alpha^\vee$, where $\langle \beta_{i,k}^\vee, \beta_{i,k+1}^\vee \rangle \neq 0$.

Lemma 6.2.2. *With $\Delta_i^\vee$ as above, suppose that $f\colon \Delta^\vee \to \mathbb{R}$ is superharmonic and is harmonic except at $\alpha^\vee$ and at $\beta_{i,1}^\vee \in \Delta_i^\vee$. Let $f_j = f|\Delta_j^\vee$. Then f_j is a linear function in the sense that $m = f_j(\beta_{j,k+1}^\vee) - f_j(\beta_{j,k}^\vee)$ is constant. If moreover $j \neq i$, then $m = f_j(\beta_{j,1}^\vee) = f_j(\alpha^\vee)/n_j$, where $n_j = \#\Delta_j^\vee$, and $f_j(\beta_{j,k}^\vee) = km$.*

Proof. The assumption that α is long implies that $n(\beta, \alpha) = -1$ for every simple coroot β which is not orthogonal to α. Thus the condition that f is harmonic except at $\beta_{i,1}^\vee$ and $\alpha^\vee$ implies that

$$2f_j(\beta_{j,k+1}^\vee) = f_j(\beta_{j,k}^\vee) + f_j(\beta_{j,k+2}^\vee)$$

for all k with $0 < k < n_j - 1$. Hence $f_j(\beta_{j,k+1}^\vee) - f_j(\beta_{j,k}^\vee)$ is constant. If in addition $j \neq i$, then $2f_j(\beta_{j,1}^\vee) = f_j(\beta_{j,2}^\vee)$ and so $f_j(\beta_{j,k+1}^\vee) - f_j(\beta_{j,k}^\vee) = f_j(\beta_{j,1}^\vee)$ for all $k < n_j - 1$. Thus $f_j(\beta_{j,k}^\vee) = kf_j(\beta_{j,1}^\vee)$. Hence, for $j \neq i$, the slope of f_j is $f_j(\alpha^\vee)/n_j$, where n_j is the cardinality of $\Delta_j^\vee$. $\square$

Proposition 6.2.3. *Suppose that R is not of type A_n. Let α be the special root, and let $f\colon \Delta^\vee \to \mathbb{R}$ be superharmonic. Then $f(\alpha^\vee)$ is the maximum value of f. More precisely, f increases (weakly) toward $\alpha^\vee$, in the sense that if $\{\beta_1^\vee, \dots, \beta_k^\vee = \alpha^\vee\} \subseteq \Delta^\vee$ has diagram a chain and is numbered so that $\langle \beta_i^\vee, \beta_{i+1}^\vee \rangle \neq 0$, then $f(\beta_i^\vee) \leq f(\beta_{i+1}^\vee)$ for $1 \leq i \leq k-1$.*

Proof. It suffices to prove this for a superharmonic function f which is harmonic except at a single coroot $\beta^\vee$. If $\beta^\vee = \alpha^\vee$, then we have seen that f is linear and increasing toward $\alpha^\vee$ on each subset $\Delta_i^\vee$ with its natural ordering.

Now suppose that $\beta^\vee \neq \alpha^\vee$. Then in particular $\beta^\vee$ does not correspond to a trivalent vertex of the Dynkin diagram, so that $\Delta^\vee - \{\beta^\vee\}$ has at most two connected components D' and D''. Let $(\Delta')^\vee$ be the set of all simple coroots in D' together with β and let $(\Delta'')^\vee$ be the set of all simple coroots in D'' together with $\beta^\vee$. We suppose that $\alpha^\vee \in (\Delta'')^\vee$. In particular, the Dynkin diagram of $(\Delta')^\vee$ is a simply laced chain (possibly consisting of a single element). Since $f' = f|(\Delta')^\vee$ is harmonic except at the endpoint $\beta^\vee$, it is linear and increases toward $\beta^\vee$. Thus the maximum value of f' is $f'(\beta^\vee)$.

The restriction $f''|(\Delta'')^\vee$ is superharmonic, and is harmonic except $\beta^\vee$, which corresponds to an end vertex. As before, we write $(\Delta'')^\vee = \bigcup_i (\Delta''_i)^\vee$, where each $(\Delta''_i)^\vee$ is a chain containing $\alpha^\vee$. We may assume that $\beta^\vee \in (\Delta''_1)^\vee$. By Lemma 6.2.2, $f''_i = f''|(\Delta''_i)^\vee$ is linear, and, for $i > 1$, f''_i increases up to $\alpha^\vee$. Moreover the slope of f''_i is $f(\alpha^\vee)/n''_i$.

First suppose that $\alpha^\vee$ is trivalent and meets $\gamma_1^\vee, \gamma_2^\vee, \gamma_3^\vee$, where $\gamma_i^\vee \in (\Delta''_i)^\vee$. The condition that f is harmonic at $\alpha^\vee$ says that $(f(\alpha^\vee)-f(\gamma_1^\vee))+(f(\alpha^\vee)-f(\gamma_2^\vee))+(f(\alpha^\vee)-f(\gamma_3^\vee)) = f(\alpha^\vee)$. For $i = 2, 3$, $f(\alpha^\vee) - f(\gamma_i^\vee)$ is the slope of f''_i, and hence is at most $f(\alpha^\vee)/2$. It follows that the slope $f(\alpha^\vee) - f(\gamma_1^\vee)$ of f''_1 is nonnegative. Thus f''_1 is also increasing toward $\alpha^\vee$. Since f' increases toward $\beta^\vee$, which is the end vertex of $(\Delta''_1)^\vee$, it follows that $f|(\Delta')^\vee \cup (\Delta''_1)^\vee$ increases toward $\alpha^\vee$.

Next suppose that R is not simply laced and that α is the long simple root which meets a short root. If α does not correspond to an end of the Dynkin diagram, then $(\Delta'')^\vee = (\Delta''_1)^\vee \cup (\Delta''_2)^\vee$, with $\beta^\vee \in (\Delta''_1)^\vee$, and $(\Delta''_2)^\vee - \{\alpha^\vee\} \neq \emptyset$. Moreover $f''_2 = f''|(\Delta''_2)^\vee$ is linear with slope $f(\alpha^\vee)/n''_2 \leq f(\alpha^\vee)/2$. If $\gamma_i^\vee \in (\Delta''_i)^\vee$ are not orthogonal to $\alpha^\vee$, then the harmonic condition at $\alpha^\vee$ says that $2f(\alpha^\vee) = m_1 f(\gamma_1^\vee) + m_2 f(\gamma_2^\vee)$ where $\{m_1, m_2\} = \{1, m\}$ with $m > 1$. Hence

$$(m_1 + m_2 - 2)f(\alpha^\vee) = m_1(f(\alpha^\vee) - f(\gamma_1^\vee)) + m_2(f(\alpha^\vee) - f(\gamma_2^\vee)),$$

and thus

$$m_1(f(\alpha^\vee) - f(\gamma_1^\vee)) \geq \left(\frac{m_2}{2} + m_1 - 2\right) f(\alpha^\vee).$$

Since $m_2/2 + m_1 - 2 \geq 0$, f''_1 is also increasing toward $\alpha^\vee$, and the proof concludes as in the trivalent case.

The remaining case is where α corresponds to an end vertex of the Dynkin diagram. In this case, if $\gamma^\vee$ is the unique simple coroot not orthogonal to $\alpha^\vee$, the harmonic condition reads:

$$(m-2)f(\alpha^\vee) = m(f(\alpha^\vee) - f(\gamma^\vee)).$$

Once again, f''_1 is increasing toward $\alpha^\vee$, and the proof concludes as before. □

Corollary 6.2.4. *If $f\colon \Delta^\vee \to \mathbb{R}$ is superharmonic, then f attains its maximum value on the coroot dual to a special root. Moreover, if α is special, then $f_{\varpi_\alpha^\vee}$ has a strict maximum at $\alpha^\vee$.*

Proof. In case R is not of type A_n, the first statement follows from Proposition 6.2.3 and the second from Lemma 6.2.2. In case R is of type A_n, the first statement is trivially true since every vertex is special, and the second again follows from Lemma 6.2.2. □

Let $\widetilde{\alpha}$ be the highest root of R and write

$$\widetilde{\alpha} = \sum_{\alpha\in\Delta} h_\alpha \alpha;$$
$$\widetilde{\alpha}^\vee = \sum_{\alpha\in\Delta} g_\alpha \alpha^\vee.$$

Corollary 6.2.5. *If α is special, then $h_\alpha = \max\{h_\beta : \beta \in \Delta\}$.*

Proof. Since $\tilde{\alpha}$ is the highest root, $n(\alpha, \tilde{\alpha}) = \alpha(\tilde{\alpha}^\vee) \geq 0$ for all $\alpha \in \Delta$, and so $f_{\tilde{\alpha}^\vee}$ is superharmonic. The result is then immediate from the previous corollary. □

It is not in general true in the non-simply laced case that, if α is a special root, then $g_\alpha = \max\{g_\beta : \beta \in \Delta\}$. However, if equality does not hold, then there is a direct argument that $\max\{g_\beta : \beta \in \Delta\} = g_\alpha + 1$.

Corollary 6.2.6. *Suppose that G is not of type A_n. Let α be special. Given $\beta_1^\vee, \dots, \beta_k^\vee = \alpha^\vee \in \Delta^\vee$, indexed so that $\langle \beta_i^\vee, \beta_{i+1}^\vee \rangle \neq 0$ for $1 \leq i \leq k-1$, then $\mu_{\beta_1} > \cdots > \mu_\alpha$.*

Proof. It suffices to show that $\mu_{\beta_1} > \mu_{\beta_2}$. By Proposition 6.2.3 applied to the superharmonic function $f_{\mu_{\beta_1}}$, we see that $1 = f_{\mu_{\beta_1}}(\beta_1^\vee) \leq f_{\mu_{\beta_1}}(\beta_2^\vee)$. Thus by Theorem 6.1.3, $\mu_{\beta_1} \geq \mu_{\beta_2}$. Since $\beta_1 \neq \beta_2$, in fact $\mu_{\beta_1} > \mu_{\beta_2}$. □

Lemma 6.2.7. *Suppose that G is not of type A_n for any n, and let $\alpha \in \Delta$ be the special vertex. Then μ_α is the unique minimally unstable Atiyah-Bott point for the trivial G-bundle over C.*

Proof. By Lemma 6.1.2 any point $\mu \in \overline{C}_0$ which is of Atiyah-Bott type for the trivial bundle and indexes a minimally unstable stratum must be of the form μ_α for some $\alpha \in \Delta$. By Corollary 6.2.6, only μ_α for α special can index a minimally unstable stratum. □

For root sysytems of type A_n, none of the μ_α are comparable, and they are all minimal elements. For root systems of type C_n or G_2, all of the order relations are accounted for by Corollary 6.2.6. In the remaining cases, not all of the order relations among the elements of the set $\{\mu_\beta : \beta \in \Delta\}$ are accounted for by Corollary 6.2.6. In the case of B_n, where the simple roots are given by $\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n$, the remaining order relations are given by: $\mu_{\alpha_n} \geq \mu_{\alpha_k}$ if and only if $k \geq n/2$. Likewise, in the case of D_n, if the simple roots are given by $\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n$, then the remaining order relations are given by: $\mu_{\alpha_{n-1}} \geq \mu_{\alpha_k}$ if and only if $k \geq n/2$, and similarly for μ_{α_n}. Of course, it is easy to work out all such inequalities for E_6, E_7, E_8, and F_4 as well.

The figures at the end of the paper give pictorial representations of the relations between the various strata corresponding to maximal parabolics in the case of a simply connected simple group.

Corollary 6.2.8. *Let G be a simply connected simple group. Let $\alpha \in \Delta$ be special. Then the stratum $\mathcal{C}_{\mu_\alpha}$ is minimally unstable. If in addition G is not of type A_n for any n, then $\mathcal{C}_{\mu_\alpha}$ is absolutely minimal in the sense that $\mathcal{C}_{\mu_\alpha} \preceq \mathcal{C}_\mu$ for all points μ of Atiyah-Bott type for the trivial bundle.* □

6.3 The non-simply connected case

Here we assume that G is simple but not necessarily simply connected. In this case as well we wish to find all minimally unstable strata. Rather than work out the theory on general principles, we shall simply specify all minimally unstable strata. In all cases, c denotes a nontrivial element of the center of the universal cover $\widetilde{G}$ of G and we assume as we may that $G = \widetilde{G}/\langle c\rangle$.

$\widetilde{G} = SL(n)$:

In this case we identify the center of $SL(n)$ with $\mathbb{Z}/n\mathbb{Z}$. Let c be a central element of order d. Then there are n/d minimally unstable strata. Their Atiyah-Bott points are μ_α/d where α is any vertex with $\varpi_\alpha(c) \equiv 1/d \pmod{\mathbb{Z}}$.

$G = SO(2n+1)$:

In this case there is a unique minimally unstable stratum. Let $\alpha_n \in \Delta$ be the unique short simple root. The Atiyah-Bott point μ for the minimally unstable stratum is $\mu_{\alpha_n}/2$.

$\widetilde{G} = Sp(2n)$:

In this case there is a unique minimally unstable stratum. Let $\alpha_n = 2e_n \in \Delta$ be the special vertex and let $\alpha_{n-1} = e_{n-1} - e_n$ be the unique short simple root which is not orthogonal to α_n. Then the Atiyah-Bott point for the minimally unstable stratum is $\mu_{\alpha_n}/2$ if n is odd and $\mu_{\alpha_{n-1}}/2$ if n is even.

$G = SO(2n)$:

In this case there are two minimally unstable strata, interchanged by the outer automorphism of $SO(2n)$. Let α_{n-1} and α_n be roots corresponding to the "ears" of the Dynkin diagram for D_n. The Atiyah-Bott points for the two minimally unstable strata are $\mu_{\alpha_{n-1}}/2$ and $\mu_{\alpha_n}/2$.

$\widetilde{G} = Spin(4n+2)$ and c has order 4:

In this case there is a unique minimally unstable stratum. Then the Atiyah-Bott point is $\mu_\beta/4$ where β is the root corresponding to an ear of the Dynkin diagram satisfying $\varpi_\beta(c) = 1/4$. Replacing c by $-c$ changes β to the other ear of the diagram.

$\widetilde{G} = Spin(4n)$ and $G \neq SO(4n)$:

In this case there is a unique minimally unstable stratum. Its Atiyah-Bott point is $\mu_{\alpha_{n-3}}/2$ where α_{n-3} is the vertex on the long arm of the Dynkin diagram next to the trivalent vertex.

$G = \mathrm{ad}\, E_6$:

In this case there is a unique minimally unstable stratum. Its Atiyah-Bott point is $\mu_\alpha/3$ where α is the unique root next to the trivalent vertex on one of the long arms of the Dynkin diagram for E_6 with $\varpi_\alpha(c) = 1/3$. Replacing c by $-c$ replaces α by the corresponding root on the other long arm of the diagram.

$G = \mathrm{ad}\, E_7$:

In this case there is a unique minimally unstable stratum. Its Atiyah-Bott point is $\mu_\alpha/2$ where α is the vertex adjacent to the trivalent vertex on the long arm.

References

[1] M. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. Roy. Soc. London A **308** (1982), 523–615.

[2] N. Bourbaki, *Groupes et Algèbres de Lie*, Chap. 4, 5, et 6, Masson, Paris, 1981.

[3] R. Friedman and J.W. Morgan, *Holomorphic principal bundles over elliptic curves*, AG/9811130.

[4] R. Friedman and J.W. Morgan, *Holomorphic principal bundles over elliptic curves II: The parabolic construction*, to appear.

[5] A. Ramanathan, *Stable principal bundles on a compact Riemann surface*, Math. Ann. **213** (1975), 129–152.

[6] S. Shatz, *The decomposition and specialization of algebraic families of vector bundles*, Comp. Math. **35** (1977), 163–187.

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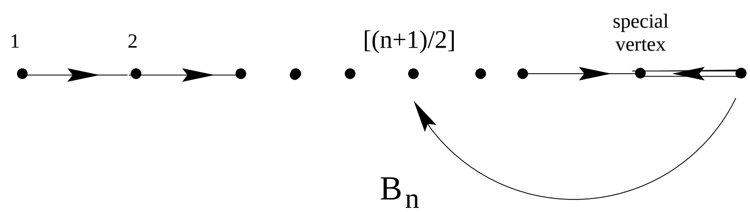
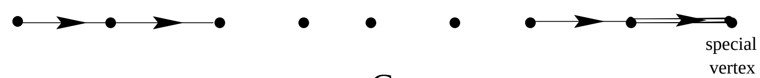
The Order Relationship on Maximal Parabolics

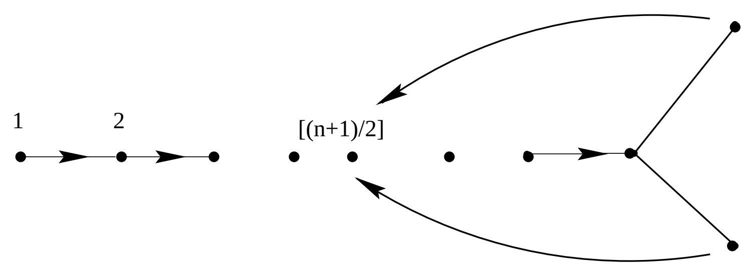
for simply connected, simple groups



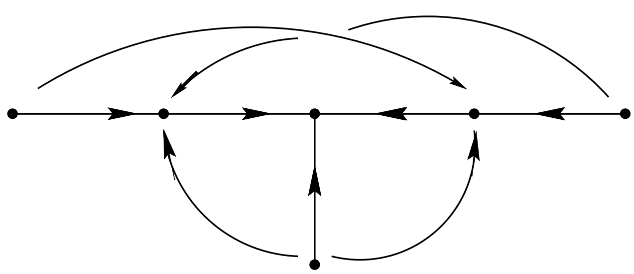
B_n



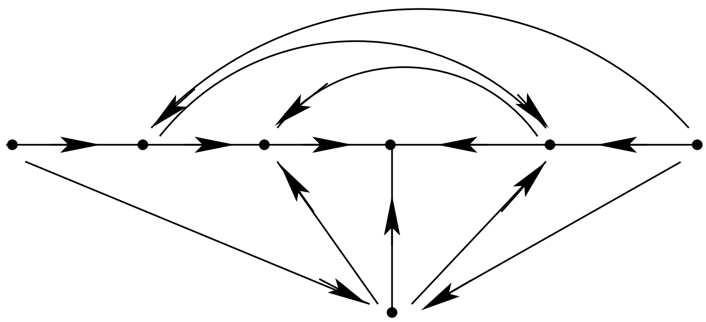
C_n



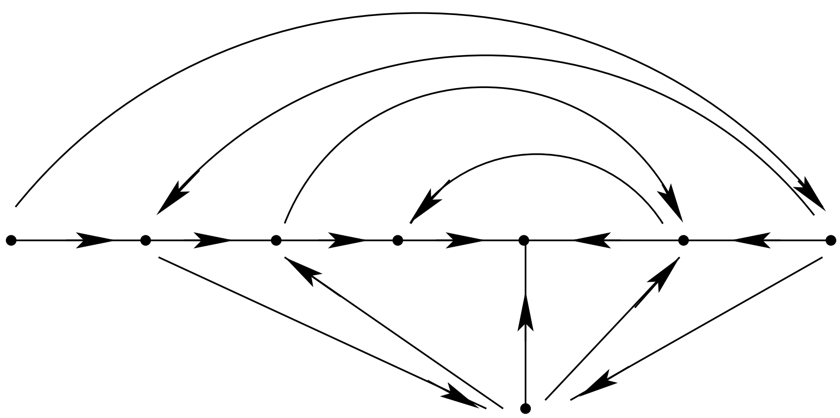
D_n



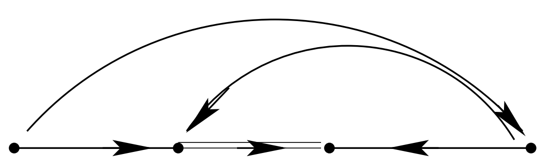
E_6



E $_7$



E $_8$



F_4